Exactly solvable models of walks:

limit distributions for counting parameters

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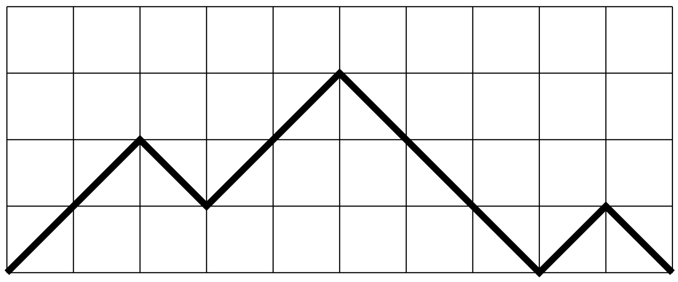
- Dyck paths: length and area
- Limit distribution for area
- Brownian excursions
- More models of walks
- More counting parameters
- Conclusion

Acknowledgements: M. Nguyễn Thế

references: `www.math.uni-bielefeld.de/~richard`

Dyck paths



Dyck path of length $2n$ ($n \in \mathbb{N}_0$)

$y : [0, 2n] \to \mathbb{R}_{\geq 0}$ (height map)

$y(0) = y(2n) = 0$, $|y(j) - y(j-1)| = 1$ ($j \in \mathbb{N}$)

$y(s)$ for non-integer s by linear extrapolation

Arch of length $2n$ ($n \in \mathbb{N}$)

Dyck path y where $y(s) > 0$ if $s \neq 0, 2n$

Combinatorial classes

$\mathcal{D}$ set of Dyck paths, $\mathcal{A}$ set of arches

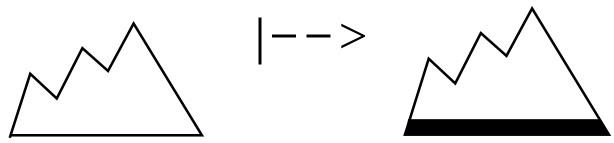
Generating functions

$w_y(x) = x^n$ weight of Dyck path y of length $2n$

$$D(x) = \sum_{d \in \mathcal{D}} w_d(x), \qquad A(x) = \sum_{a \in \mathcal{A}} w_a(x)$$

Combinatorial constructions

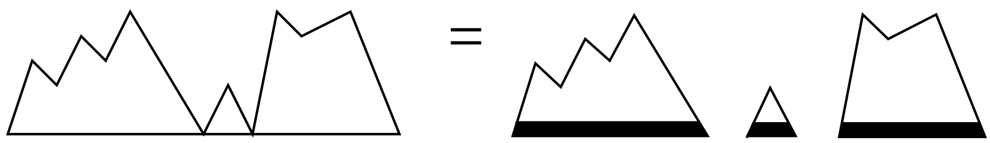
1) Path lifting:



Dyck path with additional bottom layer $\widehat{=}$ arch

$$A(x) = \sum_{d \in \mathcal{D}} w_{\bar{d}}(x) = \sum_{d \in \mathcal{D}} x w_d(x) = x D(x)$$

2) Arch decomposition:



Dyck path $\widehat{=}$ ordered sequence of arches

$$\begin{aligned} D(x) &= \sum_{k \geq 0} \sum_{(a_1,\ldots,a_k) \in \mathcal{A}^k} w_{(a_1,\ldots,a_k)}(x) \\ &= \sum_{k \geq 0} \sum_{(a_1,\ldots,a_k) \in \mathcal{A}^k} w_{a_1}(x) \cdot \ldots \cdot w_{a_k}(x) \\ &= \sum_{k \geq 0} \left(\sum_{a \in \mathcal{A}} w_a(x) \right)^k = \frac{1}{1 - A(x)} \end{aligned}$$

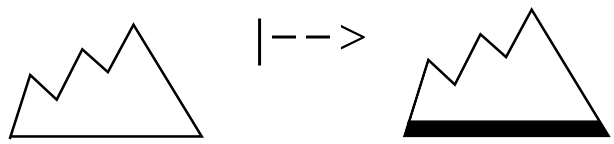
(length additive w.r.t. sequence construction)

solution: $D(x) = (1 - \sqrt{1-4x})/(2x) = \sum_{n \geq 0} C_n x^n$
($C_n = \binom{2n}{n}/(n+1)$ Catalan numbers)

Dyck paths: length and area

$D(x,q)=\sum_{d\in\mathcal{D}} w_d(x,q)$, $A(x,q)=\sum_{a\in\mathcal{A}} w_a(x,q)$
$w_y(x,q)=x^n q^m$ weight of path y of length $2n$, area m

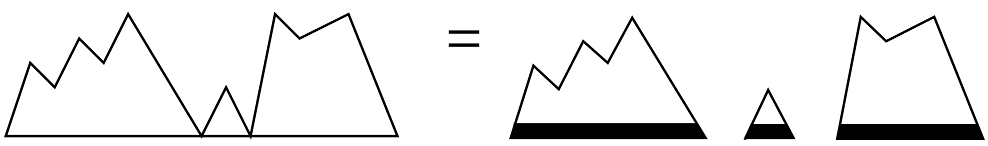
1) Path lifting:



$w_{\bar{d}}(x,q)=x^{n+1}q^{m+2n+1}=xq(xq^2)^n q^m = xq w_d(xq^2,q)$

$$A(x,q)=xqD(xq^2,q)$$

2) Arch decomposition:



$$D(x,q)=\frac{1}{1-A(x,q)}$$

(length, area additive w.r.t. sequence construction)

$$D(x,q)=\frac{1}{1-xqD(xq^2,q)}$$

q-quadratic functional equation

linearisation yields explicit expression for $D(x,q)$
as ratio of q-deformed Bessel functions

Probabilistic description

$D(x,q) = \sum_{n,m} p_{n,m} x^n q^m$ generating function

$p_{n,m}$ # Dyck paths of length $2n$, area m

Select paths of length $2n$ uniformly at random

$$\mathbb{P}(\widetilde{X}_n = m) = \frac{p_{n,m}}{\sum_m p_{n,m}}$$

Mean area of a Dyck path of length $2n$:

$$\mathbb{E}[\widetilde{X}_n] = \frac{\sum_m m\, p_{n,m}}{\sum_m p_{n,m}} = \frac{[x^n]\, \frac{\partial}{\partial q} D(x,q)\Big|_{q=1}}{[x^n] D(x,1)}$$

Differentiate functional equation w.r.t. q

$$\frac{\partial}{\partial q} D(x,q)\Big|_{q=1} = \frac{1-2x+\sqrt{1-4x}}{2x(1-4x)} = \frac{1}{4}s^{-1} - s^{-1/2} + \mathcal{O}(s^0)$$

$s = 1/4 - x$

Asymptotic coefficient extraction:

$$[x^n]\frac{1}{(x_c - x)^\gamma} = \frac{1}{x_c^\gamma \Gamma(\gamma)} x_c^{-n} n^{\gamma-1}\left(1 + \mathcal{O}(n^{-1})\right)$$

$$\mathbb{E}[\widetilde{X}_n] = \frac{4^n}{C_n}\left(1 + \mathcal{O}(n^{-\frac{1}{2}})\right) = \sqrt{\pi} n^{3/2}\left(1 + \mathcal{O}(n^{-\frac{1}{2}})\right)$$

Mean area scales with length as $n^{3/2}$!

Higher (factorial) moments

$$\mathbb{E}[(\widetilde{X}_n)_k] = \frac{\sum_m (m)_k p_{n,m}}{\sum p_{n,m}} = \frac{[x^n]\,\frac{\partial^k}{\partial q^k} D(x,q)\Big|_{q=1}}{[x^n] D(x,1)}$$

$(a)_k = a(a-1) \cdot \ldots \cdot (a-k+1)$

Differentiate functional equation w.r.t. q

$$\frac{1}{k!}\,\frac{\partial^k}{\partial q^k} D(x,q)\Big|_{q=1} = \frac{f_k}{(x_c - x)^{\gamma_k}}\left(1 + \mathcal{O}((x_c - x)^{1/2})\right)$$

γ_k exponent: $\gamma_k = \frac{3k-1}{2}$

f_k amplitude: $\gamma_{k-1} f_{k-1} + \sum_{0 \le l \le k} f_l f_{k-l} = 0, \quad f_0 = -4$

Asymptotic coefficient extraction:

$$[x^n]\frac{1}{(x_c - x)^{\gamma}} = \frac{1}{x_c^{\gamma}\Gamma(\gamma)} x_c^{-n} n^{\gamma-1}\left(1 + \mathcal{O}(n^{-1})\right)$$

$$\mathbb{E}[(\widetilde{X}_n)_k] = \frac{k!}{f_0 x_c^{\gamma_k - \gamma_0}} \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} f_k n^{3k/2}\left(1 + \mathcal{O}(n^{-1/2})\right)$$

Normalised random variable $X_n = \widetilde{X}_n / n^{3/2}$

$$\mathbb{E}[(X_n)_k] = \frac{k!}{f_0 x_c^{\gamma_k - \gamma_0}} \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} f_k \left(1 + \mathcal{O}(n^{-1/2})\right)$$

(Factorial) moments asymptotically constant!

Limit distributions

Factorial moments and ordinary moments of X_n coincide asymptotically!

$$m_k := \lim_{n\to\infty} \mathbb{E}[X_n^k] = \frac{k!}{f_0 x_c^{\gamma_k-\gamma_0}} \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} f_k$$

Carleman condition $\sum_k (m_k)^{-1/k} = +\infty$ satisfied

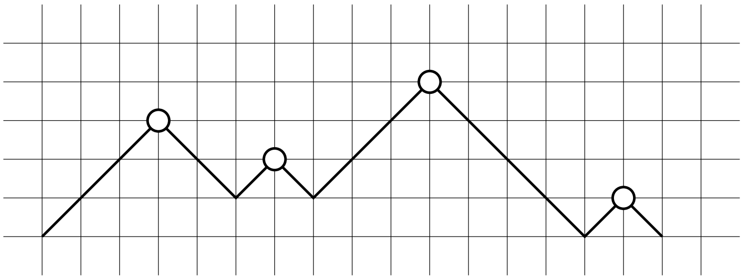
implies existence and uniqueness of a limit distribution with moments m_k

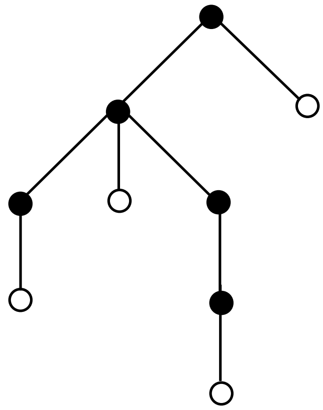
Airy distribution

appears in different contexts: area under a Brownian excursion, staircase polygon area, distribution for path length in trees

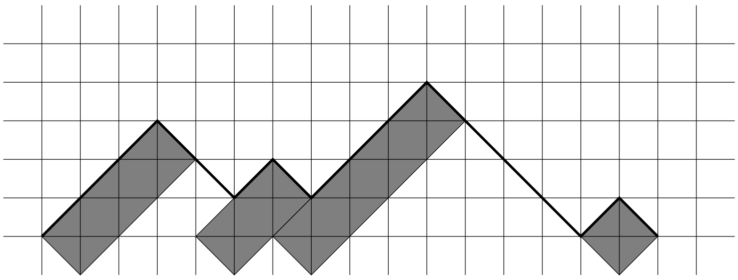
Combinatorics of trees and polygons

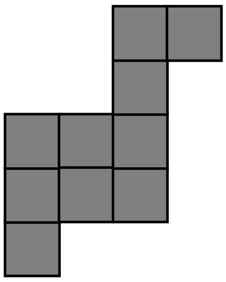
bijection to Catalan trees:
depth first search leads to Dyck paths





bijection to staircase polygons:
height and relative position of polygon columns coded by Dyck path

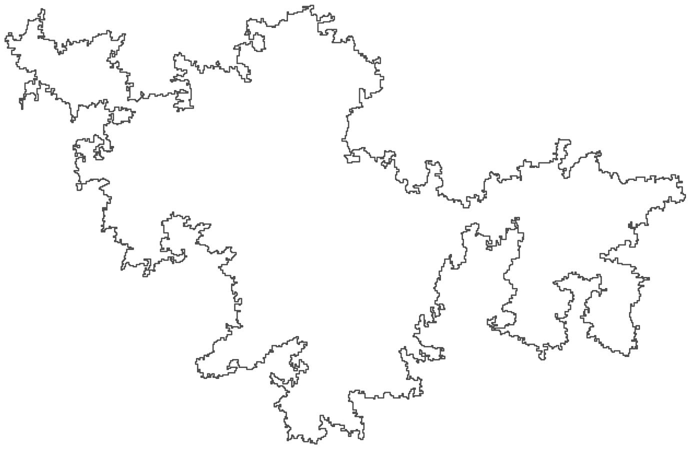




Same types of equations for walk models, tree models, and polygon models!

$\sum$ peak heights $= \sum$ leave depths $= \sum$ column heights

Self-avoiding polygons



Numerical observation: Area is Airy distributed!

moment extrapolation using exact enumeration data

(R, Guttmann, Jensen 01)

Independent of underlying lattice

tested for square, triangular, hexagonal lattice

Reason?

Rooted SAPs $G^{(r)}(x,q) = x\frac{d}{dx}G(x,q)$ display square-root singularity – same exponents as for Dyck paths

Field theoretical derivation of scaling function (Cardy 01)

Scaling limit of SAPs (if it exists) described by Brownian excursions (Werner et al. 02)

Method of dominant balance

$$
\begin{aligned}
D(x,q) &= \sum_{k\geq 0} \frac{1}{k!} \frac{\partial^k}{\partial q^k} D(x,q)\bigg|_{q=1} (q-1)^k \\
&\sim 2 + \sum_{k\geq 0} \frac{f_k}{(x_c - x)^{\gamma_k}} (q-1)^k \\
&= 2 + (x_c - x)^{1/2} \sum_{k\geq 0} f_k (-1)^k \left(\frac{1-q}{(x_c-x)^{3/2}}\right)^k \\
&= 2 + (x_c - x)^{1/2} F\left(\frac{1-q}{(x_c-x)^{3/2}}\right)
\end{aligned}
$$

$F(\epsilon) = \sum_{k\geq 0} (-1)^k f_k \epsilon^k$ scaling function

functional equation yields DE for $F(\epsilon)$:

introduce variables s, ϵ via $x = x_c - s^2$, $q = 1 - \epsilon s^3$

insert scaling form and expand up to order s^3

$$
\epsilon \left(\frac{1}{2} F(\epsilon) - \frac{3}{2} \epsilon F'(\epsilon)\right) + F(\epsilon)^2 = 16
$$

$$
\begin{aligned}
F(\epsilon) &= \epsilon^{1/3} \frac{2^{4/3} \mathrm{Ai}'(2^{4/3} \epsilon^{-2/3})}{\mathrm{Ai}(2^{4/3} \epsilon^{-2/3})} \\
&= -4 - \frac{1}{4}\epsilon + \frac{5}{128}\epsilon^2 - \frac{15}{1024}\epsilon^3 + \mathcal{O}(\epsilon^4)
\end{aligned}
$$

$\mathrm{Ai}(z) = \frac{1}{\pi} \int_0^\infty \cos(t^3/3 + tz) \mathrm{d}t$ Airy function

Moment generating function

Random variable X defined via moments

$$\mathbb{E}[X^k] = \frac{k!}{f_0 x_c^{\gamma_k - \gamma_0}} \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} f_k$$

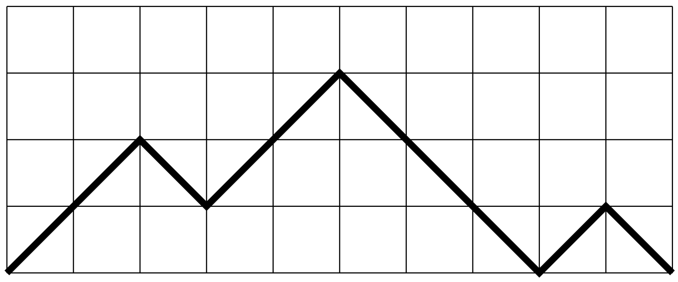
$\mathbb{E}\left[e^{-tX}\right]$ moment generating function of X

Relation to scaling function $F(\epsilon) = \sum (-1)^k f_k \epsilon^k$

$$\int_0^\infty e^{-\alpha t} \left(\mathbb{E}\left[e^{-t^{3/2} X}\right] - 1\right) \frac{\mathrm{d}t}{t^{3/2}} = \frac{\sqrt{\pi}}{2} \alpha^{1/2} \left(F(8\alpha^{-3/2}) - F(0)\right)$$

Laplace transform (use $\Gamma(\gamma) = \int_0^\infty e^{-t} t^{\gamma-1} \mathrm{d}t$)

Brownian excursions



$\mathcal{D}_{2n}$ set of Dyck paths of length $2n$
$\widetilde{Y}_n(s)$ random variable of height ($0 \leq s \leq 2n$)

$$Y_n(t) = \frac{1}{\sqrt{2n}} \widetilde{Y}_n(2nt) \qquad (0 \leq t \leq 1)$$

(mean height scales with length as $n^{1/2}$)

Theorem (e.g. Aldous 91)
$(Y_n(t))_{m \in \mathbb{N}}$ sequence of stochastic processes

$$Y_n(t) \xrightarrow{d} e(t) \qquad (n \to \infty)$$

$e(t)$ standard Brownian excursion of duration 1

What is known about excursion moments

$$X^{(k)} = \int_0^1 e^k(s) \mathrm{d}s?$$

Louchard's theorem

$V \geq 0$ symmetric, piecewise continuous
$e(s)$ standard Brownian excursion

$X = \int_0^1 V(e(s))\mathrm{d}s$ random variable

$\mathbb{E}\left[e^{-tX}\right]$ moment generating function

characterised via Laplace transform:

g_α solution bounded at infinity of

$$-\frac{1}{2}u''(x) + (\alpha + V(x))u(x) = 0$$

Theorem (Louchard 84)

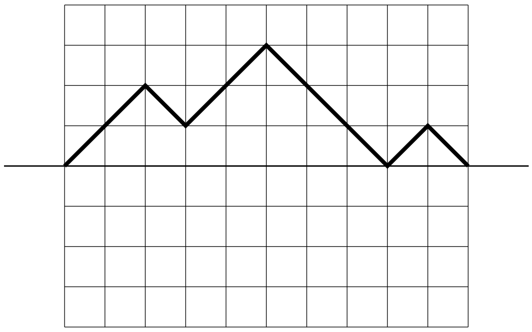
$$\int_0^\infty (e^{-\alpha t} - 1)\mathbb{E}\left[e^{-t^{3/2}\int_0^1 V(e(s))\mathrm{d}s}\right]\frac{\mathrm{d}t}{t^{3/2}} = -\left(\frac{g'_\alpha(0)}{g_\alpha(0)} + \lim_{\alpha \to 0}\frac{g'_\alpha(0)}{g_\alpha(0)}\right)$$

Explicit solution for $V(x) = x^a$,
where $a = -2, -1, 0, 1, 2$.

$a = 1$: area under excursion ($g_\alpha(0)$ Airy function)
$a = 2$: moment of inertia ($g_\alpha(0)$ Gamma function)

Dyck paths and bilateral Dyck paths

Dyck path: non-neg. RW starting and ending in $y = 0$



ordered sequence of Dyck paths with additional bottom layer

$$D(x, q) = \frac{1}{1 - x^2 q D(qx, q)}$$

counted by (total) length and area

bilateral Dyck path: RW starting and ending in $y = 0$

ordered sequence of positive or negative Dyck paths with additional bottom layer

$$B(x, q) = \frac{1}{1 - 2x^2 q D(qx, q)}$$

counted by length and *absolute* area

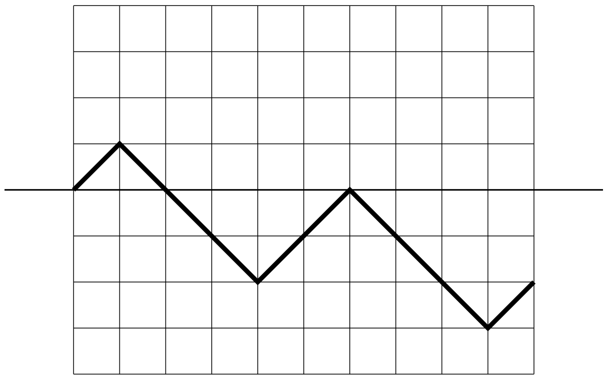
Meanders and random walks

Meander: non-neg. RW starting in $y = 0$

Dyck path or Dyck path, followed by meander with additional bottom layer

$$M(x,q) = D(x,q)(1 + xqM(qx,q))$$

Random walk



bilateral Dyck path or bilateral Dyck paths, followed by a positive or negative meander with additional bottom layer

$$R(x,q) = B(x,q)(1 + 2xqM(qx,q))$$

counted by length and *absolute* area

Same techniques as above can be applied in order to derive limit distributions for bilateral Dyck paths, Meanders and random walks

Relations between scaling functions

$$\epsilon\left(\frac{1}{2}F^{(D)}(\epsilon)-\frac{3}{2}\epsilon F^{(D)'}(\epsilon)\right)+F^{(D)}(\epsilon)^2=16$$

$$F^{(B)}(\epsilon)F^{(D)}(\epsilon)+2=0$$

$$\epsilon\left(\frac{1}{2}F^{(M)}(\epsilon)+\frac{3}{2}\epsilon F^{(M)'}(\epsilon)\right)+F^{(D)}(\epsilon)F^{(M)}(\epsilon)=4$$

$$F^{(R)}(\epsilon)=F^{(B)}(\epsilon)F^{(M)}(\epsilon)$$

Limit distributions

$$\int_0^\infty \left(e^{-\alpha t}-1\right)\mathbb{E}\left[e^{-\sqrt{2}t^{3/2}X^{(D)}}\right]\frac{\mathrm{d}t}{t^{3/2}}=2\sqrt{\pi}\left(\frac{\mathrm{Ai}(\alpha)}{\mathrm{Ai}'(\alpha)}-\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)}\right)$$

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\left[e^{-\sqrt{2}t^{3/2}X^{(B)}}\right]\frac{\mathrm{d}t}{t^{1/2}}=-\sqrt{\pi}\frac{\mathrm{Ai}(\alpha)}{\mathrm{Ai}'(\alpha)}$$

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\left[e^{-\sqrt{2}t^{3/2}X^{(M)}}\right]\frac{\mathrm{d}t}{t^{1/2}}=-\sqrt{\pi}\frac{2\int_0^\alpha \mathrm{Ai}(s)\mathrm{d}s-1}{3\mathrm{Ai}(\alpha)}$$

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\left[e^{-\sqrt{2}t^{3/2}X^{(R)}}\right]\mathrm{d}t=\frac{2\int_0^\alpha \mathrm{Ai}(s)\mathrm{d}s-1}{3\mathrm{Ai}'(\alpha)}$$

Dyck paths: k-th moments of height

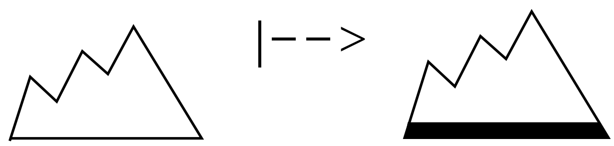
Dyck path y of length $2n$

$$n_k = \sum_{i=0}^{2n} y^k(i)$$

weight $w_y(\boldsymbol{u}) = u_0^{2n} u_1^{n_1} \cdot \ldots \cdot u_M^{n_M}$

$D(\boldsymbol{u}) = \sum_{d \in \mathcal{D}} w_d(\boldsymbol{u})$, $A(\boldsymbol{u}) = \sum_{a \in \mathcal{A}} w_a(\boldsymbol{u})$

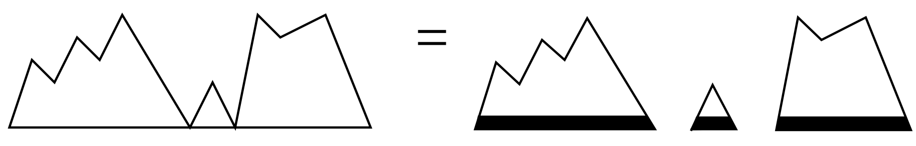
1) Path lifting:



$$A(\boldsymbol{u}) = u_0^2 u_1 \cdot \ldots \cdot u_M D(\boldsymbol{v}(\boldsymbol{u}))$$

$$v_k(\boldsymbol{u}) = \prod_{l=k}^{M} u_l^{\binom{l}{k}}$$

2) Arch decomposition:



$$D(\boldsymbol{u}) = \frac{1}{1 - A(\boldsymbol{u})}$$

(sequence construction: length, height moments additive)

q-quadratic functional equation

Dyck paths: k-th moments of height

method of dominant balance can be applied

Theorem

$e(t)$ standard Brownian excursion of duration 1
k-th excursion moment $X_k = \int_0^1 e^k(s)\, \mathrm{d}s$

$$\mathbb{E}[X_1^{k_1} \cdot \ldots \cdot X_M^{k_M}] = \boldsymbol{k}! \frac{\sqrt{2\pi}}{\Gamma(\gamma_{\boldsymbol{k}})} 2^{\gamma_{\boldsymbol{k}}} \frac{f_{\boldsymbol{k}}}{2},$$

where $\gamma_{\boldsymbol{k}} = -1/2 + \sum_{i=1}^{M}(1 + i/2)k_i$.

Recursion for $f_{\boldsymbol{k}}$ ($\boldsymbol{k} \neq \boldsymbol{0}$, $\boldsymbol{e}_i$ unit vectors)

$$f_{\boldsymbol{k}} = \frac{1}{8}\gamma_{\boldsymbol{k}-\boldsymbol{e}_1} f_{\boldsymbol{k}-\boldsymbol{e}_1} + \sum_{i=1}^{M-1} \frac{i+1}{4}(k_i + 1) f_{\boldsymbol{k}-\boldsymbol{e}_{i+1}+\boldsymbol{e}_i} + \frac{1}{8} \sum_{\substack{\boldsymbol{l}\neq\boldsymbol{0},\boldsymbol{l}\neq\boldsymbol{k} \\ \boldsymbol{0}\leq\boldsymbol{l}\leq\boldsymbol{k}}} f_{\boldsymbol{l}} f_{\boldsymbol{k}-\boldsymbol{l}},$$

boundary conditions: $f_0 = -4$, $f_k = 0$ if $k_j < 0$ for some j

No closed form solutions for $M > 2$.

Corresponding results for Brownian bridges, meanders, and Brownian motion

Conclusion

General method for deriving limit distributions

reproduces (known) area laws of corresponding stochastic objects

yields (new) moment recurrences for a number of other counting parameters
(e.g. excursion moments)

results apply in a more general context
(q-functional equations)

includes models of trees and polygons