Disproving things is easier when you know they are false

Experimenting with two conjectures on pattern-avoiding permutations

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Work with...

My collaborators in this work are

- Murray Elder — University of St Andrews, Scotland.
- Mike Zabrocki — York University, Canada.
- Paul Westcott — some badly run bank in Melbourne.
- Michael Albert — University of Otago, New Zealand.

Outline

My first lesson from supervisor

- I learnt lots of maths from my PhD supervisors.
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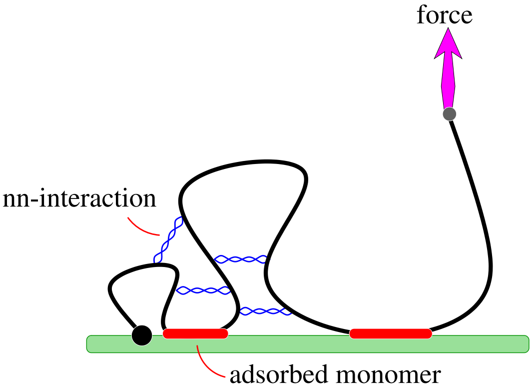
Lesson context

- I usually work on

Lattice models of polymers



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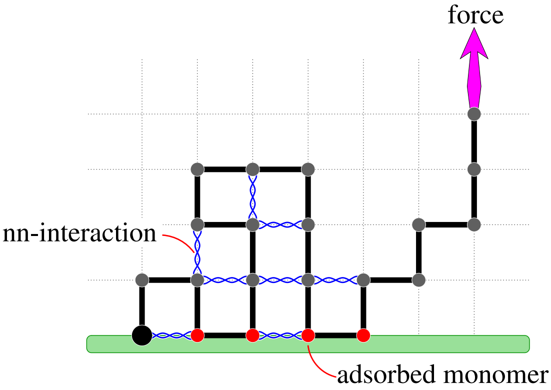
Lattice models of polymers

- Self-avoiding walks and related objects
- Related and more easily solved models



Lesson context

The sorts of results I have looked for

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Results

- Find generating functions.

If c_n is the number of objects of size n, the generating function is

$$f(z) = \sum_{n \geq 0} c_n z^n$$

Guess using Maple etc. . .

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Results

- Find generating functions.
- Find growth constants and free energies

If c_n is the number of objects of size n, the growth constant is

$$\mu = \lim_{n \to \infty} c_n^{1/n}$$

Guess using numerics or simulations

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Results

- Find generating functions.
- Find growth constants and free energies
- Some physics and scaling.

How does the system behave? Are there phase transitions?
Study the g.f. or use numerics and simulations.

Lesson context

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Results

- Find generating functions.
- Find growth constants and free energies
- Some physics and scaling.
- What type of solutions do unsolved problems have?

Is there a polynomial time algorithm to find c_n?
Is the g.f. rational, algebraic or D-finite?

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Results

- Find generating functions.
- Find growth constants and free energies
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- What type of solutions do unsolved problems have?

- Almost all my work has been *on lattice*.

A recent diversion into permutations

- In 2004 I visited Murray Elder in St-Andrews.
- He introduced me to some data-sorting problems.
- Similar objects turn up in algebraic combinatorics.

Pattern-avoiding permutations

Some definitions

- What is a pattern?
- How do we know if a permutation avoids it?

Some definitions

A pattern

A pattern of length k is a permutation of $\{1, 2, \ldots, k\}$.

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A permutation σ of length n contains the pattern τ of length k if we can delete all but k elements of σ and reduce it to get τ.

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Reducing a vector

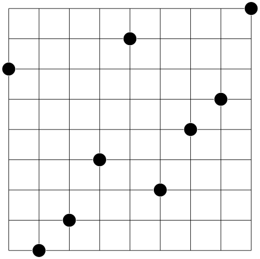
A vector v is *reduced* by replacing

- its smallest element with 1,
- its second smallest with 2,

and so on.

Examples

The permutation 712483569 contains the pattern 1324



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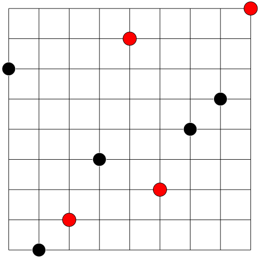

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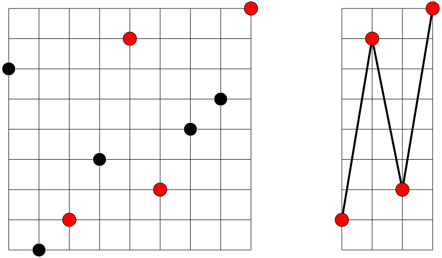

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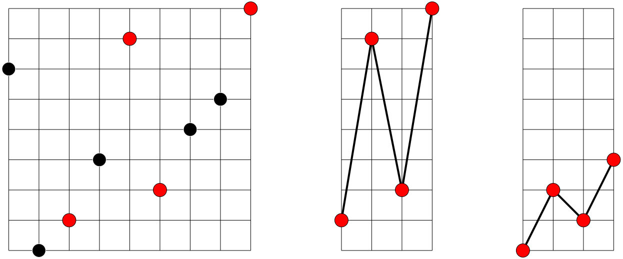

Examples

The permutation 712483569 contains the pattern 1324

- The permutation contains the subsequence 2839
- This reduces to 1324

The permutation 769384521 avoids 1324

- It contains no subsequence that reduces to 1324.
- Checking by hand is laborious.
- The computer does a good job!

The core problem

The generating function

- Let $S_n(\tau)$ be # permutations of length n that avoid τ.
- Let $P_\tau(z)$ be the generating function $\sum_{n\geq 0} S_n(\tau) z^n$.

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The generating function

- Let $S_n(\tau)$ be $\#$ permutations of length n that avoid τ.
- Let $P_\tau(z)$ be the generating function $\sum_{n \geq 0} S_n(\tau) z^n$.

- Core problem $=$ find either of these for a given τ.
- We would be happy with
  - closed expression for the coefficients
  - formula for the generating function
  - a recurrence
  - asymptotics
  - growth constant

Very few results

- (Un)fortunately the problem seems to be very hard.

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- $\tau = 12 \ldots k$ — Gessel.
- and a few more.

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- and a few more.
- **But not** 1324 and 4231.

- Very difficult since pattern avoidance is a non-local condition.
- There are some conjectures. . .

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Only very recently proved — Marcos and Tardos.

Another conjecture

Arratia-Bóna Conjecture

Let τ be a permutation of length k, then

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- Based on small number of known $\mu(\tau)$
- No one had done (serious) numerics!
- Getting series data is hard — μ is big!

Nature of the generating function

Differentiably finite

A power series $f(z)$ is *differentiably finite* if

- it satisfies a DE of the form

$$q_d(z)f^{(d)}(z) + \cdots + q_1(z)f'(z) + q_0(z)f(z) = 0$$

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- Many solved combinatorial models have D-finite solutions.
- Many unsolved ones probably do not!

Yet another conjecture

Zeilberger-Noonan Conjecture

For a given τ, the generating function

$$P_\tau(z) = \sum_{n\geq 0} S_n(\tau) z^n$$

is a differentiably finite power series.

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is a differentiably finite power series.

- Based on small number of known gf
- Not all of these gf are algebraic — Gessel, Bousquet-Mélou.
- Getting series data for GFUN is hard.

The first thing I do with a new problem. . .

First thing to do

- Write some code and generate some numbers.

Second thing to do

- Play with the numbers

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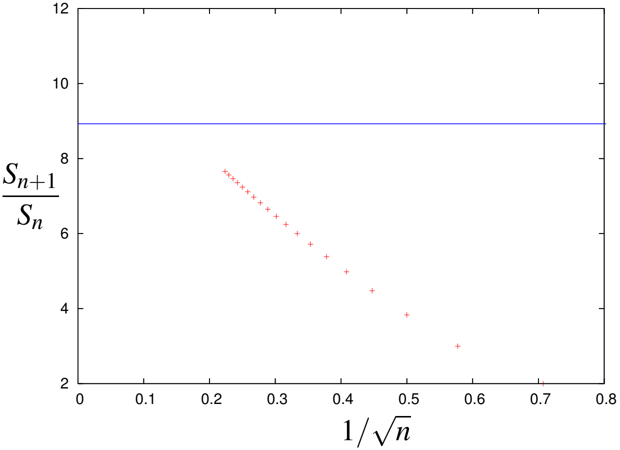
Second thing to do

- Play with the numbers

- First unsolved pattern is 1324 ($\equiv 4231$).
- Brute-force enumeration is slow — $\mu \leq 9$? (by conjecture)
- Marinov and Radoičić found a much faster way.
- Unfortunately it is still exponential time.

Plot the enumeration data

- Get numbers from Sloan and play with them:



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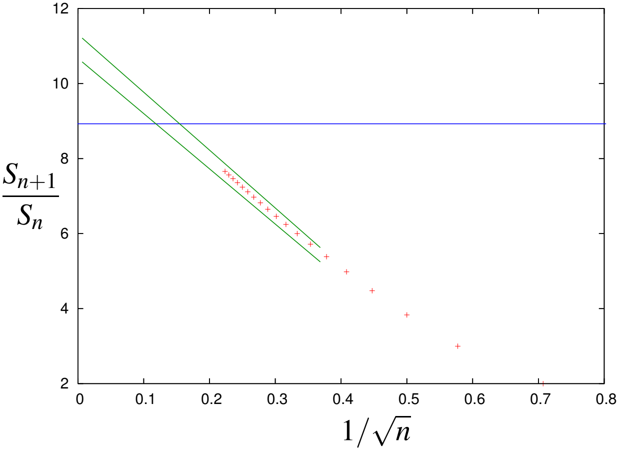

- Ratio should $\to \mu$ and its going way past 9.
- The conjecture looks pretty shaky!

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The growth constant of a large subset $\implies$ lower bound on μ.

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- Grow 23154 by $1 \to \circ 1$
- Restrict insertions to first few positions.

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- Mike, Murray and I proved the rational form.
- Can do similarly for any given pattern.
- More general result — Albert, Atkinson & Ruškuc.

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- Not good bounds — $\mu \geq 3$.
- Try restricted growth instead.
- Cannot use the Guttmann-Enting solvability test?

Guttmann-Enting solvability test

Rewrite the gf as

$$F(z,t) = \sum_{k\geq 0} H_k(z) t^k$$

and look at $H_k(z)$.

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Guttmann + Enting observed

- The $H_k(z)$ are usually rational.
- Solved models $\implies$ the H_k have a small number of poles.
- Unsolved models $\implies$ the H_k have more and more poles.

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This can be made more rigorous

Bousquet-Mélou

- Let $\mathcal{S}$ be the set of singularities of the H_k.
- If $\mathcal{S}$ is dense then $F(z,t)$ is not D-finite.

Arratia-Bóna conjecture for 1324-avoiders

Count big subsets by restricting growth

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Count big subsets by restricting growth

Insert only in first k positions

- Checking for pattern
  — only the first k entries of the permutation are important.
- Hence finite-state automata with $k!$ states.
- Work and think harder — 4^k states.
- Dominant eigenvalue gives lower bound for μ.

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Need 2 tricks for efficient memory use:

- combinatorial trick for simple description of the automata states.
- a real c++ programmer — Paul Westcott.

So close. . .

Very memory hungry

- Memory growth is 4^k.
- We used 16Gb (credit to Tony Guttmann).
- Got $\mu > 8.7$.

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But all was not lost. . .

So close. . . We have a disproof!

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Less memory hungry

- Michael Albert had a different growth method.
- Restrict the number of “slots”.
- Memory growth rate is $(1+\sqrt{2})^k$
- Gives $\mu(1324) = \mu(4231) > 9.45$.

Arratia owes us US$100!
Albert, Elder, Rechnitzer, Westcott & Zabrocki — $20 each.

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Look in more detail at 1324-avoiders with k descents

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The denominator connection

- The generating function is

$$G_k(z) = \frac{\text{some polynomial}}{(1-z)^{d_k}}$$

- The first few d_k are $1, 4, 8, 12, 17, 22, 27, \ldots$

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- DS numbers grow superlinearly.
- Superlinear denominator growth $\implies$ not D-finite.
- A way of attacking the Zeilberger-Noonan conjecture!

Davenport-Schinzel sequences

DS sequences

A word on s symbols is a Davenport-Schinzel sequence if

- No adjacencies: $w_i \neq w_{i+1}$.
- No alternating subsequences: *ababa*.

The s^{th} DS number is the max length of such a sequence.

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$s = 1$	a	1
$s = 2$	a, b, a, b	4
$s = 3$	a, b, a, c, a, c, b, c	8

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Faster than linear

The maximum length of a DS sequence on s symbols is

- $O(s\alpha(s))$ where $\alpha =$ inverse Ackermann.
- Faster than linear (but only just).

Linking 1324-avoiders to Davenport-Schnizel

Simplifying things

- Look at $G_k\left(\frac{z}{1+z}\right)$ — simple positive polynomial.
- Counts "squashed" 1324-avoiders with k descents. "squashed" means $\sigma_{i+1} \neq \sigma_i + 1$.
- Degree of polynomial $=$ max length $= d_k$.

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- Degree of polynomial = max length = d_k.

There is then a simple mapping:

squashed 1324-avoiders $\mapsto$ a subset of DS-sequences

Since it is a subset we do not have superlinearity *yet*!

Where we are at

- $\exists$ constructive proof of the superlinear growth of DS sequences.
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- Trying to alter this proof for 1324-avoiders.

- This would show that the 2-variable g.f. is not D-finite.
- Unfortunately does not disprove the Zeilberger-Noonan conjecture for the 1-variable g.f.
- But does make it less likely.
- In fact, Zeilberger no longer believes his conjecture.

Where we are going. . .

Trying hard to prove

Elder-Rechnitzer-Zabrocki Conjecture

For 1324-avoiders, the two variable generating function

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Done some preliminary work on

Growth constant classification

- Use FlatPERM to do approximate enumeration.
- Estimate $\mu(\tau)$ for different τ.
- What makes a pattern hard to avoid?

Approximate enumeration

- Since we cannot find μ exactly we would like to estimate it.

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- The algorithm we use is based on the Rosenbluth[2] method.
- It is called FlatPERM — a major developer is Thomas Prellberg.

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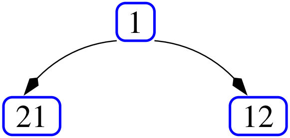

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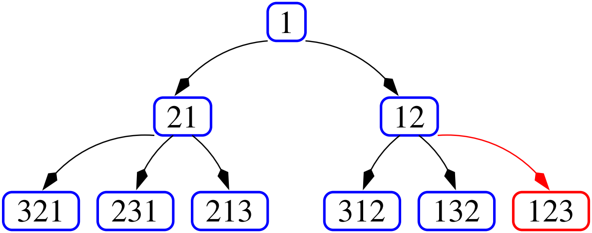

If we are looking at 123-avoiders.

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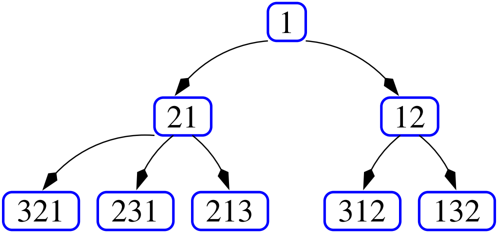

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Rosenbluth sampling

Random path on the tree

- Start at the root.
- Choose a child of current node uniformly at random.
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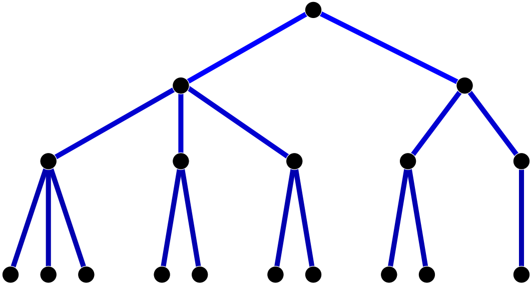
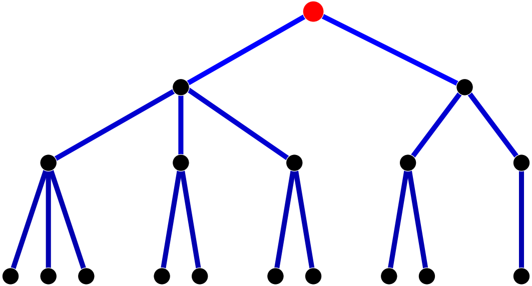
- The nodes at a given depth are not chosen with uniform probability.
- But this allows us to estimate the # nodes at a given depth.

Probability of paths



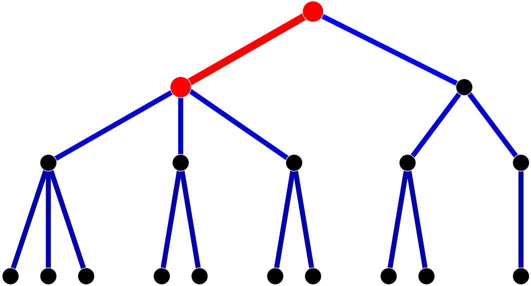
$$\Pr(\text{ node }) =$$

Probability of paths



$$\Pr(\text{ node }) = 1$$

Probability of paths



$$\Pr(\text{ node }) = 1 \cdot \frac{1}{2}$$

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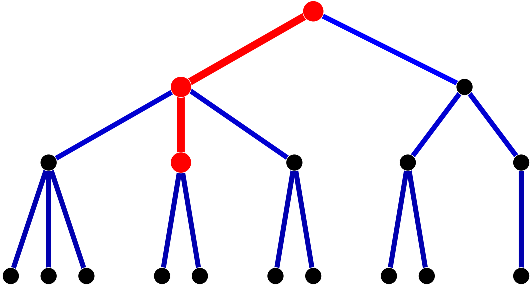
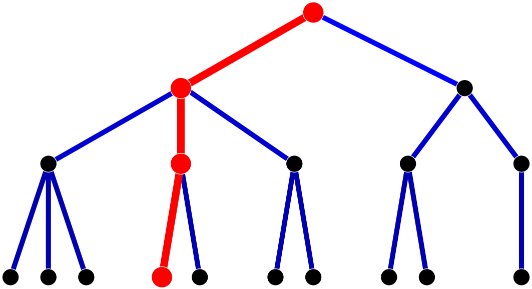


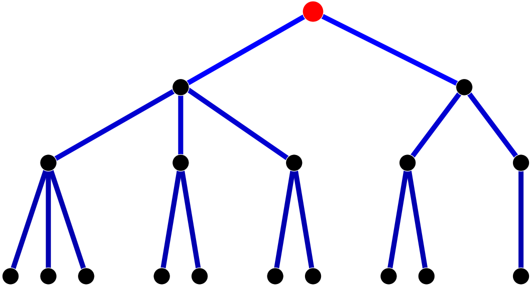
$$\Pr(\text{ node }) = 1 \cdot \frac{1}{2} \cdot \frac{1}{3}$$

Probability of paths



$$\Pr(\text{ node }) = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{12}$$

Probability of paths



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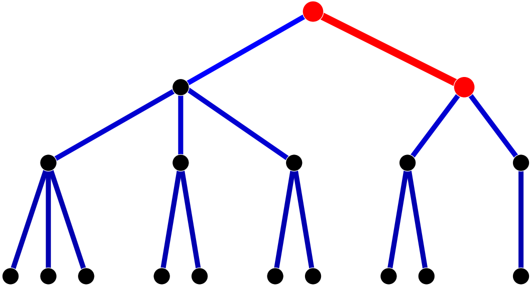

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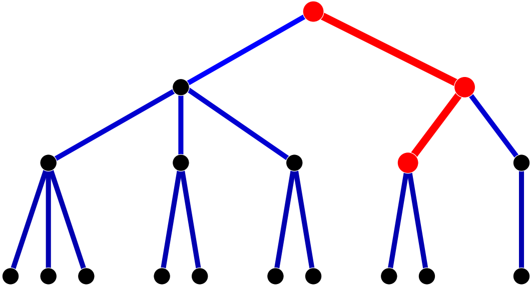

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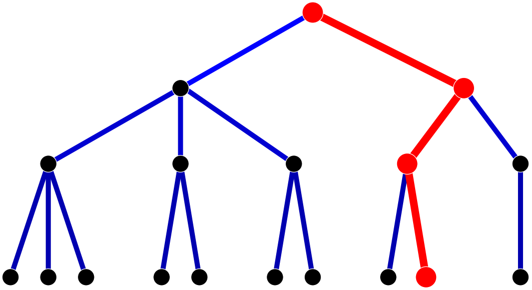

$$\Pr(\text{ node }) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Weights

Atmosphere and weight

- Let $a(\text{node})$ = its number of children.

$$w(\text{node}) = \begin{cases} 1 & \text{node} = \text{root} \\ a(\text{parent})w(\text{parent}) & \text{otherwise} \end{cases}$$

This then gives

$$\Pr(\text{node}) = 1/w(\text{node})$$

and

$$\langle w(\text{node}) \rangle = \#\text{nodes}$$

Problems with this

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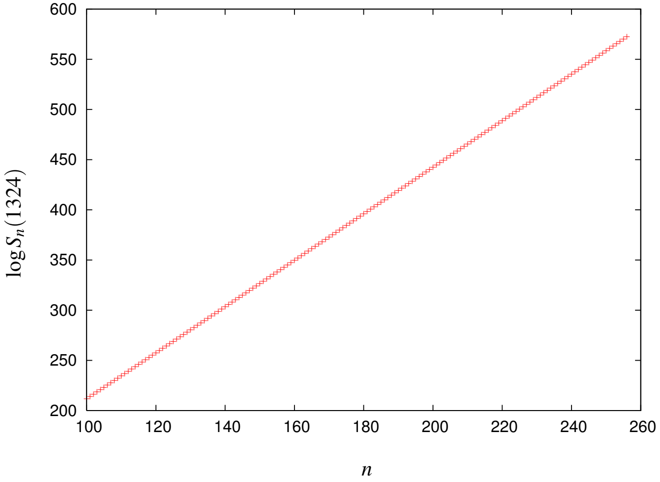
$$\text{RR} \implies \text{PERM} \implies \text{FlatPERM}$$

Tentative results



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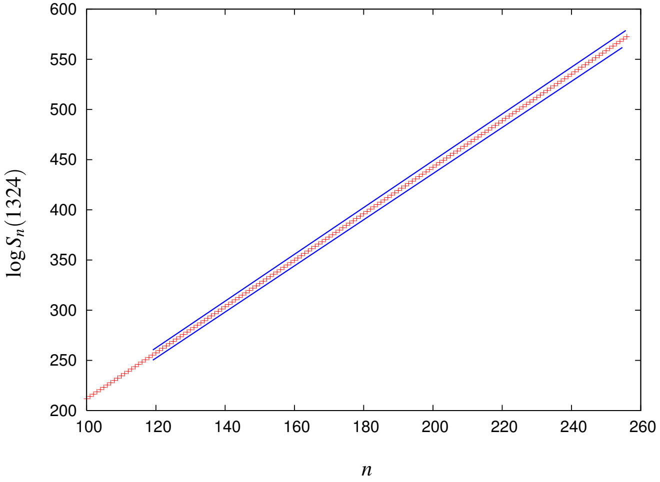

Gives $\mu(1324) \approx 10.3(2)$.

Thanks for listening

Questions?