The solution to the prize question

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December 11, 2007

Question: Find

$$f(x) = \int_0^1 \frac{x}{\sqrt{2\pi\alpha^3(1-\alpha)}} \exp\left(-\frac{x^2}{2\alpha(1-\alpha)}\right) d\alpha$$

Answer: Clearly, $f(0) = 0$ and $f(-x) = -f(x)$, so that we only need to consider $x > 0$.

The integral simplifies after a suitable substitution. Finding the correct substitution is the major obstacle here, so we try to indicate how one could have obtained it.

The argument of the exponential function contains the term

$$\frac{1}{2\alpha(1-\alpha)}$$

which is minimal at $\alpha = 1/2$ (it reaches the value 2), and tends to infinity as α tends to 0 or 1. Knowing that one can solve integrals such as $\int_{-\infty}^{\infty} \exp(-t^2) dt = \sqrt{\pi}$, we would like to attempt a substitution which exploits this. If we write

$$\frac{1}{2\alpha(1-\alpha)} = 2(1+u^2)$$

and solve for α, we are led to the substitution

$$\alpha = \frac{1}{2}\left(1 + \frac{u}{\sqrt{1+u^2}}\right) .$$

Using this substitution for the integral, $f(x)$ indeed simplifies considerably to

$$f(x) = \exp(-2x^2)\,\frac{2x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 - \frac{u}{\sqrt{1+u^2}}\right) \exp(-2x^2u^2)du \ ,$$

which indicates that this substitution was a reasonable one.

Splitting the integral now into two integrals gives

$$f(x) = \exp(-2x^2)\,(f_1(x) - f_2(x))$$

with

$$\begin{aligned} f_1(x) &= \frac{2x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-2x^2u^2)du \ , \\ f_2(x) &= \frac{2x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{u}{\sqrt{1+u^2}} \exp(-2x^2u^2)du \ . \end{aligned}$$

We see that $f_2(x) = 0$, as the integrand is an odd function of u, so that $f(x) = \exp(-2x^2)f_1(x)$. After substitution of

$$u = \frac{t}{\sqrt{2}x}$$

into $f_1(x)$, we arrive at

$$f_1(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2)dt = 1$$

as $\int_{-\infty}^{\infty} \exp(-t^2)dt = \sqrt{\pi}$. The final answer covering all possible values of x is therefore

$$f(x) = \mathrm{sgn}(x)\exp(-2x^2) \ .$$