MAS115 Calculus I 2007-2008

Problem sheet for exercise class 8

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

(*) Problem 1: [2007 exam questions]

Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)dt \ ?$$

a. h is a twice-differentiable function of x.
b. h and dh/dx are both continuous.
c. The graph of h has a horizontal tangent at $x = 1$.
d. h has a local maximum at $x = 1$.
e. h has a local minimum at $x = 1$.
f. The graph of h has an inflection point at $x = 1$.
g. The graph of dh/dx crosses the x-axis at $x = 1$.

Problem 2: Sometimes it helps to reduce the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. Practice this on

$$\int \sqrt{1 + \sin^2(x-1)}\,\sin(x-1)\cos(x-1)dx \ .$$

a. $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$
b. $u = \sin(x - 1)$, followed by $v = 1 + v^2$
c. $u = 1 + \sin^2(x - 1)$

Problem 3: Suppose that $f(x)$ is positive, continuous, and increasing over the interval $[a, b]$. By interpreting the graph of f show that

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) \ .$$

Extra: Prove that

$$\int_0^x \left(\int_0^u f(t)dt \right) du = \int_0^x f(u)(x-u)du \ .$$

(Hint: Express the integral on the right hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

Thomas Prellberg, November 2007

Problem 1:

(a) True: by Part 1 of the Fundamental Theorem of Calculus, $h'(x) = f(x)$. Since f is differentiable for all x, h has a second derivative for all x.

(b) True: they are continuous because they are differentiable.

(c) True, since $h'(1) = f(1) = 0$.

(d) True, since $h'(1) = 0$ and $h''(1) = f'(1) < 0$.

(e) False, since $h''(1) = f'(1) < 0$.

(f) False, since $h''(x) = f'(x) < 0$ never changes sign.

(g) True, since $h'(1) = f(1) = 0$ and $h'(x) = f(x)$ is a decreasing function of x (because $f'(x) < 0$).

Problem 2:

(a) Let $u = x - 1 \Rightarrow du = dx$; $v = \sin u \Rightarrow dv = \cos u\, du$; $w = 1 + v^2 \Rightarrow dw = 2v\, dv \Rightarrow \frac{1}{2}\, dw = v\, dv$

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int \sqrt{1+\sin^2 u}\,\sin u\cos u\,du = \int v\sqrt{1+v^2}\,dv$$

$$= \int \tfrac{1}{2}\sqrt{w}\,dw = \tfrac{1}{3}w^{3/2} + C = \tfrac{1}{3}\left(1+v^2\right)^{3/2} + C = \tfrac{1}{3}\left(1+\sin^2 u\right)^{3/2} + C = \tfrac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$$

(b) Let $u = \sin(x-1) \Rightarrow du = \cos(x-1)\,dx$; $v = 1 + u^2 \Rightarrow dv = 2u\,du \Rightarrow \frac{1}{2}\,dv = u\,du$

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int u\sqrt{1+u^2}\,du = \int \tfrac{1}{2}\sqrt{v}\,dv = \int \tfrac{1}{2}v^{1/2}\,dv$$

$$= \left(\tfrac{1}{2}\left(\tfrac{2}{3}\right)v^{3/2}\right) + C = \tfrac{1}{3}v^{3/2} + C = \tfrac{1}{3}\left(1+u^2\right)^{3/2} + C = \tfrac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$$

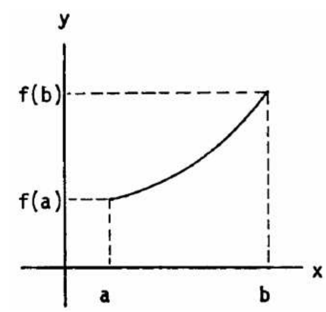
(c) Let $u = 1 + \sin^2(x-1) \Rightarrow du = 2\sin(x-1)\cos(x-1)\,dx \Rightarrow \frac{1}{2}\,du = \sin(x-1)\cos(x-1)\,dx$

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int \tfrac{1}{2}\sqrt{u}\,du = \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2}\right) + C$$

$$= \tfrac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$$

Problem 3:

The first integral is the area between f(x) and the x-axis over $a \le x \le b$. The second integral is the area between f(x) and the y-axis for $f(a) \le y \le f(b)$. The sum of the integrals is the area of the larger rectangle with corners at (0, 0), (b, 0), (b, f(b)) and (0, f(b)) minus the area of the smaller rectangle with vertices at (0, 0), (a, 0), (a, f(a)) and (0, f(a)). That is, the sum of the integrals is $bf(b) - af(a)$.



Extra:

The derivative of the left side of the equation is: $\frac{d}{dx}\left[\int_0^x \left[\int_0^u f(t)\,dt\right] du\right] = \int_0^x f(t)\,dt$; the derivative of the right side of the equation is: $\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\int_0^x f(u)\,x\,du - \frac{d}{dx}\int_0^x u\,f(u)\,du$

$$= \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\int_0^x u\,f(u)\,du = \int_0^x f(u)\,du + x\left[\frac{d}{dx}\int_0^x f(u)\,du\right] - xf(x) = \int_0^x f(u)\,du + xf(x) - xf(x)$$

$= \int_0^x f(u)\,du$. Since each side has the same derivative, they differ by a constant, and since both sides equal 0 when $x = 0$, the constant must be 0. Therefore, $\int_0^x \left[\int_0^u f(t)\,dt\right] du = \int_0^x f(u)(x-u)\,du$.