MAS115 Calculus I 2007-2008

Problem sheet for exercise class 7

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Problem 1:(*) a. Evaluate

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \ldots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5 dx$$

and evaluating the integral.

b. Evaluate

$$\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{n^4} .$$

Problem 2: Which formula is not equivalent to the other two?

a. $\sum_{j=2}^{4} \frac{(-1)^{j-1}}{j-1}$

b. $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$

c. $\sum_{l=-1}^{1} \frac{(-1)^{l}}{l+2}$

Problem 3: L'Hopital's rule does not help with the following limits. Find them some other way:

a. $\lim_{x\to\infty} \frac{\sqrt{x+5}}{\sqrt{x}+5}$

b. $\lim_{x\to\infty} \frac{2x}{x+7\sqrt{x}}$

Extra: Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, fing $h(10)$.

Thomas Prellberg, November 2007

Problem 1:

a.
Let $f(x) = x^5$ on $[0, 1]$. Partition $[0, 1]$ into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is increasing on $[0, 1]$, $U = \sum_{j=1}^{\infty} \left(\frac{j}{n}\right)^5 \left(\frac{1}{n}\right)$ is the upper sum for $f(x) = x^5$ on $[0, 1]$ $\Rightarrow \lim_{n\to\infty} \sum_{j=1}^{\infty} \left(\frac{j}{n}\right)^5 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \ldots + \left(\frac{n}{n}\right)^5\right] = \lim_{n\to\infty} \left[\frac{1^5+2^5+\ldots+n^5}{n^6}\right]$
$= \int_0^1 x^5\,dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}$

b.
Let $f(x) = x^3$ on $[0, 1]$. Partition $[0, 1]$ into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is increasing on $[0, 1]$, $U = \sum_{j=1}^{\infty} \left(\frac{j}{n}\right)^3 \left(\frac{1}{n}\right)$ is the upper sum for $f(x) = x^3$ on $[0, 1]$ $\Rightarrow \lim_{n\to\infty} \sum_{j=1}^{\infty} \left(\frac{j}{n}\right)^3 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \ldots + \left(\frac{n}{n}\right)^3\right] = \lim_{n\to\infty} \left[\frac{1^3+2^3+\ldots+n^3}{n^4}\right]$
$= \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

Problem 2:

$$\text{(a)}\quad \sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1} = \frac{(-1)^{2-1}}{2-1} + \frac{(-1)^{3-1}}{3-1} + \frac{(-1)^{4-1}}{4-1} = -1 + \frac{1}{2} - \frac{1}{3}$$

$$\text{(b)}\quad \sum_{k=0}^{2} \frac{(-1)^{k}}{k+1} = \frac{(-1)^{0}}{0+1} + \frac{(-1)^{1}}{1+1} + \frac{(-1)^{2}}{2+1} = 1 - \frac{1}{2} + \frac{1}{3}$$

$$\text{(c)}\quad \sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2} = \frac{(-1)^{-1}}{-1+2} + \frac{(-1)^{0}}{0+2} + \frac{(-1)^{1}}{1+2} = -1 + \frac{1}{2} - \frac{1}{3}$$

(a) and (c) are equivalent; (b) is not equivalent to the other two.

Problem 3:

$$\text{(a)}\quad \lim_{x\to\infty} \frac{\sqrt{x+5}}{\sqrt{x}+5} = \lim_{x\to\infty} \frac{\frac{\sqrt{x+5}}{\sqrt{x}}}{\frac{\sqrt{x}+5}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{5}{x}}}{1+\frac{5}{\sqrt{x}}} = \frac{1}{1} = 1$$

$$\text{(b)}\quad \lim_{x\to\infty} \frac{2x}{x+7\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{2x}{x}}{\frac{x+7\sqrt{x}}{x}} = \lim_{x\to\infty} \frac{2}{1+7\sqrt{\frac{1}{x}}} = \frac{2}{1+0} = 2$$

Extra:

$h(x) = f^2(x) + g^2(x) \Rightarrow h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2\left[f(x)f'(x) + g(x)g'(x)\right] = 2\left[f(x)g(x) + g(x)(-f(x))\right]$
$= 2 \cdot 0 = 0$. Thus $h(x) = c$, a constant. Since $h(0) = 5$, $h(x) = 5$ for all x in the domain of h. Thus $h(10) = 5$.