MAS115 Calculus I 2007-2008

Problem sheet for exercise class 6

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

(*) Problem 1: Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

Problem 2: Sketch the graph of $f(x) = \frac{x^3}{3x^2+1}$.

Problem 3: The sum of two non-negative numbers is 20. Find the numbers

a. if the product of one number and the square root of the other is to be as large as possible.

b. if one number plus the square root of the other is to be as large as possible.

Extra: The family of straight lines $y = ax + b$ (a, b arbitrary constants) can be characterised by the relation $y'' = 0$. Find a similar relation satisfied by the family of all circles

$$(x - h)^2 + (y - h)^2 = r^2 ,$$

where h and r are arbitrary constants.

Thomas Prellberg, November 2007

Problem 1:

Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.4).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative Test. We will see in Step 6 that both are absolute extrema as well.
4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2}$$

Expanding numerator

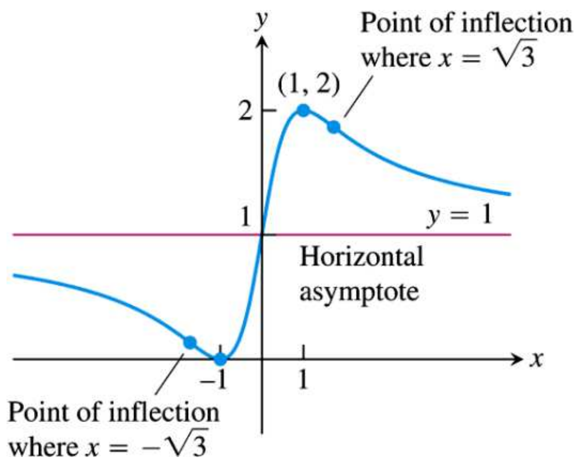
$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}.$$

Dividing by x^2

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

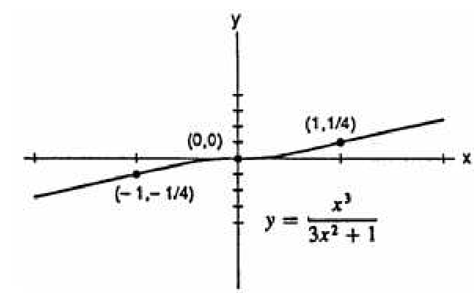
Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).

7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■



Problem 2:

When $y = \frac{x^3}{3x^2+1}$, then $y' = \frac{3x^2(3x^2+1) - x^3(6x)}{(3x^2+1)^2}$
$= \frac{3x^2(x^2+1)}{(3x^2+1)^2}$ and
$y'' = \frac{(12x^3+6x)(3x^2+1)^2 - 2(3x^2+1)(6x)(3x^4+3x^2)}{(3x^2+1)^4}$
$= \frac{6x(1-x)(1+x)}{(3x^2+1)^3}$. The curve is rising on $(-\infty, \infty)$ so there are no local extrema. The curve is concave up on $(-\infty, -1)$ and $(0, 1)$, and concave down on $(-1, 0)$ and $(1, \infty)$. There are points of inflection at $x = -1$, $x = 0$, and $x = 1$.



Problem 3:

(a) Maximize $f(x) = \sqrt{x}(20 - x) = 20x^{1/2} - x^{3/2}$ where $0 \le x \le 20 \Rightarrow f'(x) = 10x^{-1/2} - \frac{3}{2}x^{1/2}$
$= \frac{20-3x}{2\sqrt{x}} = 0 \Rightarrow x = 0$ and $x = \frac{20}{3}$ are critical points; $f(0) = f(20) = 0$ and $f\left(\frac{20}{3}\right) = \sqrt{\frac{20}{3}}\left(20 - \frac{20}{3}\right)$
$= \frac{40\sqrt{20}}{3\sqrt{3}} \Rightarrow$ the numbers are $\frac{20}{3}$ and $\frac{40}{3}$.

(b) Maximize $g(x) = x + \sqrt{20 - x} = x + (20 - x)^{1/2}$ where $0 \le x \le 20 \Rightarrow g'(x) = \frac{2\sqrt{20-x} - 1}{2\sqrt{20-x}} = 0$
$\Rightarrow \sqrt{20 - x} = \frac{1}{2} \Rightarrow x = \frac{79}{4}$. The critical points are $x = \frac{79}{4}$ and $x = 20$. Since $g\left(\frac{79}{4}\right) = \frac{81}{4}$ and $g(20) = 20$, the numbers must be $\frac{79}{4}$ and $\frac{1}{4}$.

Extra:

We have that $(x - h)^2 + (y - h)^2 = r^2$ and so $2(x - h) + 2(y - h)\frac{dy}{dx} = 0$ and $2 + 2\frac{dy}{dx} + 2(y - h)\frac{d^2y}{dx^2} = 0$ hold. Thus $2x + 2y\frac{dy}{dx} = 2h + 2h\frac{dy}{dx}$, by the former. Solving for h, we obtain $h = \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}$. Substituting this into the second equation yields $2 + 2\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} - 2\left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right) = 0$. Dividing by 2 results in $1 + \frac{dy}{dx} + y\frac{d^2y}{dx^2} - \left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right) = 0$.