MAS115 Calculus I 2007-2008

Problem sheet for exercise class 5

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Problem 1: [2007 exam questions]

a. State the definition of the derivative of the function $f(x)$ with respect to the variable x.

b. Given
$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0 \quad \text{and} \quad \lim_{x\to 0}\frac{\sin x}{x} = 1\ ,$$
differentiate from first principles $f(x) = \cos x$.

(*) Problem 2: Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

Problem 3: Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for the answer.

Extra: Suppose that a function f satisfies the following conditions for all real values of x and y:

i. $f(x + y) = f(x)f(y)$.

ii. $f(x) = 1 + xg(x)$, where $\lim_{x\to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

Thomas Prellberg, October 2007

Problem 1

[exam question]

a)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{if the limit exists}$$

b)
$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\cos x (\cos h - 1)}{h} - \frac{\sin x \sin h}{h} \right) = -\cos x \overset{0}{\lim_{h \to 0} \frac{1-\cos h}{h}} - \sin x \overset{1}{\lim_{h \to 0} \frac{\sin h}{h}}$$

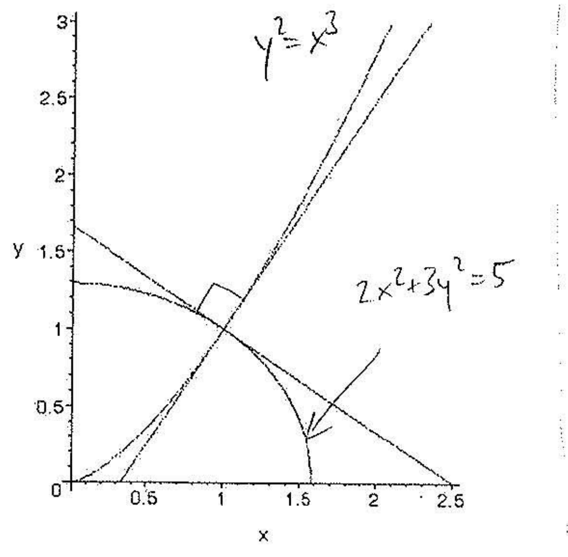
Problem 2

(*)

For the curve $y = \sqrt{x}$, we have $y' = \cancel{\lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{(x+h) - x}{(\sqrt{x+h} + \sqrt{x})h}}$ $\cancel{= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}} = \frac{1}{2\sqrt{x}}$. Suppose $(a, \sqrt{a})$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}$ which must also equal $\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1)$ $\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

Problem 3

$2x^2 + 3y^2 = 5 \Rightarrow 4x + 6yy' = 0 \Rightarrow y' = -\frac{2x}{3y} \Rightarrow y'|_{(1,1)} = -\frac{2x}{3y}\Big|_{(1,1)} = -\frac{2}{3}$ and $y'|_{(1,-1)} = -\frac{2x}{3y}\Big|_{(1,-1)} = \frac{2}{3}$; also, $y^2 = x^3 \Rightarrow 2yy' = 3x^2 \Rightarrow y' = \frac{3x^2}{2y} \Rightarrow y'|_{(1,1)} = \frac{3x^2}{2y}\Big|_{(1,1)} = \frac{3}{2}$ and $y'|_{(1,-1)} = \frac{3x^2}{2y}\Big|_{(1,-1)} = -\frac{3}{2}$. Therefore the tangents to the curves are perpendicular at $(1, 1)$ and $(1, -1)$ (i.e., the curves are orthogonal at these two points of intersection).



Extra

From the given conditions we have $f(x + h) = f(x)\,f(h)$, $f(h) - 1 = hg(h)$ and $\lim_{h \to 0} g(h) = 1$. Therefore, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)\,f(h) - f(x)}{h} = \lim_{h \to 0} f(x) \left[\frac{f(h) - 1}{h}\right] = f(x) \left[\lim_{h \to 0} g(h)\right] = f(x) \cdot 1 = f(x)$ $\Rightarrow f'(x) = f(x)$ and $f'(x)$ exists at every value of x.