MAS115 Calculus I 2007-2008

Problem sheet for exercise class 4

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Problem 1: **Continuity.**

(*) a. Can $f(x) = x(x^2-1)/|x^2-1|$ be extended to be continuous at $x=1$ or $x=-1$? Give reasons for your answers.

b. For what value of a is [2007 exam questions]

$$f(x) = \begin{cases} x^2 - 1\,, & x < 3 \\ 2ax\,, & x \geq 3 \end{cases}$$

continuous at every x?

Problem 2: **Limits and continuity.** Which of the following statements are true and which false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

a. If f is continuous at a, then so is $|f|$.

b. If $|f|$ is continuous at a, then so is f.

Problem 3: **The Intermediate Value Theorem.** [2007 exam questions]

a. What are the hypotheses and conclusions of the Intermediate Value Theorem?

b. Using the Intermediate Value Theorem, explain why the equation

$$\cos x = x$$

has at least one solution.

Extra: **A function continuous at only one point.** Let

$$f(x) = \begin{cases} x\,, & \text{if } x \text{ is rational} \\ 0\,, & \text{if } x \text{ is irrational.} \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

Thomas Prellberg, October 2007

Problem 1 a)

At $x = -1$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x(x^2-1)}{|x^2-1|}$

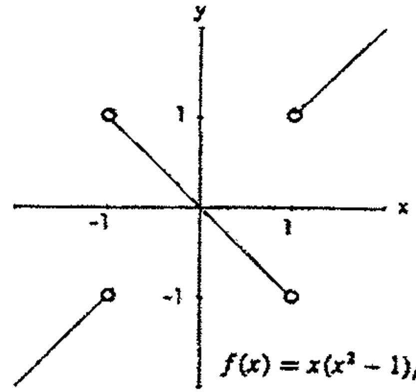
$= \lim_{x \to -1^-} \frac{x(x^2-1)}{x^2-1} = \lim_{x \to -1^-} x = -1$, and

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to -1^+} \frac{x(x^2-1)}{-(x^2-1)}$

$= \lim_{x \to -1} (-x) = -(-1) = 1$. Since

$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

$\Rightarrow \lim_{x \to -1} f(x)$ does not exist, the function f <u>cannot</u> be extended to a continuous function at $x = -1$.



$f(x) = x(x^2 - 1)/|x^2 - 1|$

At $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to 1^-} \frac{x(x^2-1)}{-(x^2-1)} = \lim_{x \to 1^-} (-x) = -1$, and

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to 1^+} \frac{x(x^2-1)}{x^2-1} = \lim_{x \to 1^+} x = 1$. Again $\lim_{x \to 1} f(x)$ does not exist so f <u>cannot</u> be extended to a continuous function at $x = 1$ either.

Problem 1(b)

$f(x)$ continuous at $x=3$ if

(i) $\lim_{x\to 3} f(x)$ exists

(ii) $f(3)$ exists

(iii) these values are equal

(i): $\lim_{x\to 3^-} f(x) = 3^2 - 1 = 8$

$\lim_{x\to 3^+} f(x) = 2a\,3 = 6a$

$\Rightarrow \lim_{x\to 3} f(x)$ exists if $8 = 6a \Rightarrow \underline{a = \frac{2}{3}}$

(ii) $f(3) = 2a\cdot 3 = 8$ exists

(iii) $\lim_{x\to 3} f(x) = f(3)$ is satisfied

Problem 2

(a) True, because $g(x) = |x|$ is continuous

$\Rightarrow g(f(x)) = |f(x)|$ is continuous as composition of continuous functions

(b) False, for example take

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

$f(x)$ is discontinuous at $x=0$ but

$|f(x)| = 1$ is continuous at $x = 0$

Problem 3

(a) Hypotheses (assumptions) are :

$f(x)$ continuous on the closed

interval $[a, b]$

Conclusions :

$f(x)$ takes on every value between

$f(a)$ and $f(b)$ on $[a, b]$

(b) Take $f(x) = \cos x - x$

Pick $a = 0, \quad b = \frac{\pi}{2}$

Then $f(0) = 1$ and $f(b) = -\frac{\pi}{2}$

Therefore $f(x)$ takes on every value

between $-\frac{\pi}{2}$ and 1 on $[0, \frac{\pi}{2}]$.

In particular, there is an $x_0 \in [0, \frac{\pi}{2}]$

with
$$0 = f(x_0) = \cos x_0 - x_0$$

Extra

(a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow |f(x) - 0| = |x - 0| < \epsilon \Leftrightarrow |x - 0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow |f(x) - 0| < \epsilon \Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Therefore, f is continuous at $x = 0$.

(b) Choose $x = c > 0$. Then within any interval $(c - \delta, c + \delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = |0 - c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c - \delta, c + \delta)$ with $|x - c| < \frac{c}{2} = \epsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0| = |x| > \frac{c}{2} = \epsilon \Rightarrow$ f is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.