MAS115 Calculus I 2007-2008

Problem sheet for exercise class 3

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

(*) Problem 1: **Two wrong statements about limits.** Show by example that the following statements are wrong.

(a) The number L is the limit of $f(x)$ as x approaches x_0 if $f(x)$ gets closer to L as x approaches x_0.

(b) The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the functions in your examples do not have the given value of L as a limit as $x \to x_0$.

Problem 2: Compute the following limits: [2007 exam questions]

(a) $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x+7}-4}$, (b) $\lim\limits_{u\to 3} \dfrac{u^3-27}{u^4-81}$, (c) $\lim\limits_{x\to 0} \dfrac{6x+6x\cos(6x)}{\sin(6x)\cos(6x)}$.

Problem 3: Use the graph of the greatest integer function $y = \lfloor x \rfloor$ to determine the limits

(a) $\lim\limits_{\theta\to 3^+} \dfrac{\lfloor\theta\rfloor}{\theta}$, $\lim\limits_{\theta\to 3^-} \dfrac{\lfloor\theta\rfloor}{\theta}$, (b) $\lim\limits_{t\to 4^+} (t - \lfloor t \rfloor)$, $\lim\limits_{t\to 4^-} (t - \lfloor t \rfloor)$.

Extra: **Roots of a quadratic equation that is almost linear.** The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1+\sqrt{1+a}}{a} , \quad r_-(a) = \frac{-1-\sqrt{1+a}}{a} .$$

(a) What happens to $r_+(a)$ as $a \to 0$? As $a \to -1^+$?

(b) What happens to $r_-(a)$ as $a \to 0$? As $a \to -1^+$?

(c) Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.

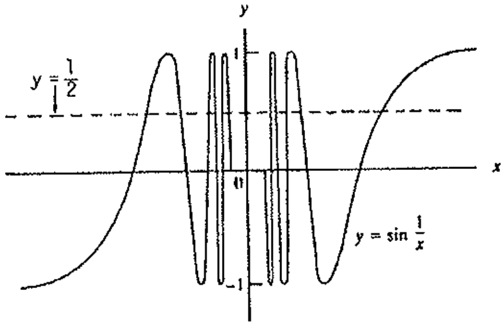
Thomas Prellberg, October 2007

(*) Problem 1 (a)

Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim_{x \to 0} f(x) = 0$, not -1. The function $f(x) = x^2$ never gets <u>arbitrarily close</u> to -1 for x near 0.

(b)

Let $f(x) = \sin x$, $L = \frac{1}{2}$, and $x_0 = 0$. There exists a value of x (namely, $x = \frac{\pi}{6}$) for which $|\sin x - \frac{1}{2}| < \epsilon$ for any given $\epsilon > 0$. However, $\lim_{x \to 0} \sin x = 0$, not $\frac{1}{2}$. The wrong statement does not require x to be arbitrarily close to x_0. As another example, let $g(x) = \sin \frac{1}{x}$, $L = \frac{1}{2}$, and $x_0 = 0$. We can choose infinitely many values of x near 0 such that $\sin \frac{1}{x} = \frac{1}{2}$ as you can see from the accompanying figure. However, $\lim_{x \to 0} \sin \frac{1}{x}$ fails to exist. The wrong statement does not require <u>all</u> values of x arbitrarily close to $x_0 = 0$ to lie within $\epsilon > 0$ of $L = \frac{1}{2}$. Again you can see from the figure that there are also infinitely many values of x near 0 such that $\sin \frac{1}{x} = 0$. If we choose $\epsilon < \frac{1}{4}$ we cannot satisfy the inequality $|\sin \frac{1}{x} - \frac{1}{2}| < \epsilon$ for all values of x sufficiently near $x_0 = 0$.



Problem 3

(a) $\lim_{\theta \to 3^+} \frac{\lfloor\theta\rfloor}{\theta} = \frac{3}{3} = 1$

$\lim_{\theta \to 3^-} \frac{\lfloor\theta\rfloor}{\theta} = \frac{2}{3}$

(b) $\lim_{t \to 4^+} (t - \lfloor t \rfloor) = 4 - 4 = 0$

$\lim_{t \to 4^-} (t - \lfloor t \rfloor) = 4 - 3 = 1$

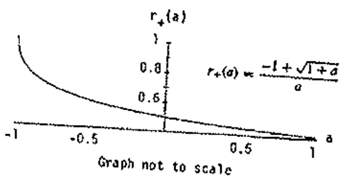
Extra

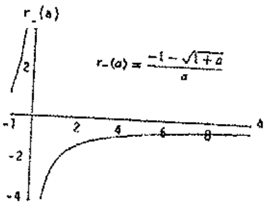
(a) At $x = 0$: $\lim_{a \to 0} r_+(a) = \lim_{a \to 0} \frac{-1+\sqrt{1+a}}{a} = \lim_{a \to 0} \left(\frac{-1+\sqrt{1+a}}{a}\right)\left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right)$
$= \lim_{a \to 0} \frac{1-(1+a)}{a(-1-\sqrt{1+a})} = \frac{-1}{-1-\sqrt{1+0}} = \frac{1}{2}$

At $x = -1$: $\lim_{a \to -1^+} r_+(a) = \lim_{a \to -1^+} \frac{1-(1+a)}{a(-1-\sqrt{1+a})} = \lim_{a \to -1} \frac{-a}{a(-1-\sqrt{1+a})} = \frac{-1}{-1-\sqrt{0}} = 1$

(b) At $x = 0$: $\lim_{a \to 0^-} r_-(a) = \lim_{a \to 0^-} \frac{-1-\sqrt{1+a}}{a} = \lim_{a \to 0^-} \left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)$
$= \lim_{a \to 0^-} \frac{1-(1+a)}{a(-1+\sqrt{1+a})} = \lim_{a \to 0^-} \frac{-a}{a(-1+\sqrt{1+a})} = \lim_{a \to 0^-} \frac{-1}{-1+\sqrt{1+a}} = \infty$ (because the denominator is always negative); $\lim_{a \to 0^+} r_-(a) = \lim_{a \to 0^+} \frac{-1}{-1+\sqrt{1+a}} = -\infty$ (because the denominator is always positive). Therefore, $\lim_{a \to 0} r_-(a)$ does not exist.

At $x = -1$: $\lim_{a \to -1^+} r_-(a) = \lim_{a \to -1^+} \frac{-1-\sqrt{1+a}}{a} = \lim_{a \to -1^+} \frac{-1}{-1+\sqrt{1+a}} = 1$





Problem 2

(a)

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x+7}-4} = \lim_{x\to 9} \frac{(x-9)\left(\sqrt{x+7}+4\right)}{\left(\sqrt{x+7}-4\right)\left(\sqrt{x+7}+4\right)}$$

$$= \lim_{x\to 9} \frac{\cancel{(x-9)}\left(\sqrt{x+7}+4\right)}{\cancel{x+7-16}} = \lim_{x\to 9}\left(\sqrt{x+7}+4\right)$$

$$= \sqrt{9+7}+4 = 8$$

(b)

$$\lim_{u\to 3} \frac{u^3-27}{u^4-81} = \lim_{u\to 3} \frac{\cancel{(u-3)}(u^2+3u+9)}{\cancel{(u-3)}(u^3+3u^2+9u+27)}$$

$$= \lim_{u\to 3} \frac{u^2+3u+9}{u^3+3u^2+9u+27} = \frac{3^2+3\cdot 3+9}{3^3+3\cdot 3^2+9\cdot 3+27}$$

$$= \frac{1}{4}$$

(c)

$$\lim_{x\to 0} \frac{6x+6x\cos 6x}{\sin 6x \cos 6x} = \lim_{x\to 0} \frac{1+\cos 6x}{\left(\frac{\sin 6x}{6x}\right)\cos 6x}$$

$$= \frac{1+\cos 0}{1\cdot\cos 0} = 2 \qquad \left(\text{used: } \lim_{t\to 0}\frac{\sin t}{t} = 1\right)$$