MAS115 Calculus I 2007-2008

Problem sheet for exercise class 2

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

Problem 1: Evaluate in terms of radicals

(*) (i) $\sin\frac{7\pi}{12}$

(ii) $\cos\frac{\pi}{12}$ [2007 exam questions]

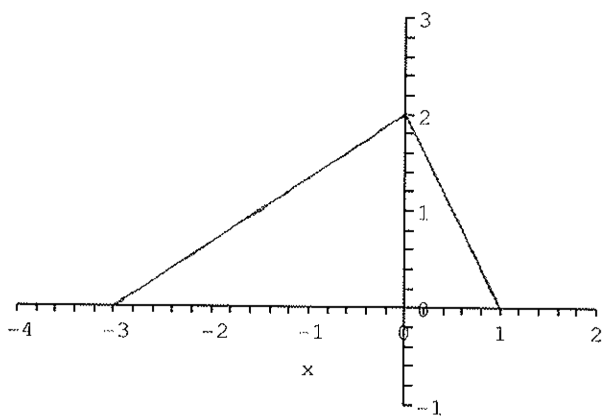
Problem 2: Find a formula for $f \circ g$ and $g \circ f$ and find the domain and range of each.

$$(a) \qquad f(x) = 2 - x^2\,, \qquad g(x) = \sqrt{x+2}$$
$$(b) \qquad f(x) = \sqrt{x}\,, \qquad g(x) = \sqrt{1-x}$$

Problem 3: Prove the identity

$$\frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

Problem 4: The graph of f is shown. Draw the graph of each function.
(a) $y = f(-x)$, (b) $y = -f(x)$, (c) $y = -2f(x+1)+1$, (d) $y = 3f(x-2)-2$.



Extra: Graph the equations (a) $|x| + |y| = 1 + x$ and (b) $y + |y| = x + |x|$.

Thomas Prellberg, October 2007

Problem 1

(*) (a) Evaluate $\sin \frac{7\pi}{12}$

(i) use $\sin^2 \alpha = \frac{1}{2}(1-\cos 2\alpha)$

$$\sin^2 \frac{7\pi}{12} = \frac{1}{2}\left(1 - \cos \frac{7\pi}{6}\right)$$

$$= \frac{1}{2}\left(1 + \cos \frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\sqrt{3}\right)$$

Sign of $\sin \frac{7\pi}{12}$: $\quad 0 \leq \frac{7}{12}\pi \leq \pi$, positive

$$\sin \frac{7\pi}{12} = \sqrt{\frac{1}{2}\left(1+\frac{1}{2}\sqrt{3}\right)} = \frac{1}{2}\sqrt{2+\sqrt{3}}$$

(ii) use $\frac{7}{12} = \frac{1}{4} + \frac{1}{3}$ & addition theorems

$$\sin \frac{7\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$= \sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3}$$

$$= \frac{\sqrt{2}}{2} \frac{1}{2} + \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

[yes, both answers are actually the same!]

(b) Evaluate $\cos \frac{\pi}{12}$ [2007 exam question]

(i) use $\cos^2 \alpha = \frac{1}{2}\left(1 + \cos 2\alpha\right)$

$$\cos^2 \frac{\pi}{12} = \frac{1}{2}\left(1 + \cos \frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)$$

sign of $\cos \frac{\pi}{12}$: $0 \leq \frac{\pi}{12} \leq \frac{\pi}{2}$, positive

$$\cos \frac{\pi}{12} = \sqrt{\frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)} = \frac{1}{2}\sqrt{2 + \sqrt{3}}$$

(ii) use $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$ & addition theorems

$$\cos \frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$$

$$= \frac{1}{2} \quad \frac{\sqrt{2}}{2} \quad + \quad \frac{\sqrt{3}}{2} \quad \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

(iii) obviously $\cos \frac{\pi}{12} = \sin \frac{7\pi}{12}$ same as in (a)

Problem 2 (a)

$(f \circ g)(x) = f(g(x)) = f(\sqrt{x+2}) = 2 - (\sqrt{x+2})^2 = -x,\ x \geq -2.$

$(g \circ f)(x) = f(g(x)) = g(2 - x^2) = \sqrt{(2-x^2)+2} = \sqrt{4-x^2}$

Domain of fog: $[-2, \infty)$.

Domain of gof: $[-2, 2]$.

Range of fog: $(-\infty, 2]$.

Range of gof: $[0, 2]$.

Problem 2 (b)

$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{1-x}\right) = \sqrt{\sqrt{1-x}} = \sqrt[4]{1-x}.$

$(g \circ f)(x) = f(g(x)) = g(\sqrt{x}) = \sqrt{1-\sqrt{x}}$

Domain of fog: $(-\infty, 1]$.

Domain of gof: $[0, 1]$.

Range of fog: $[0, \infty)$.

Range of gof: $[0, 1]$.

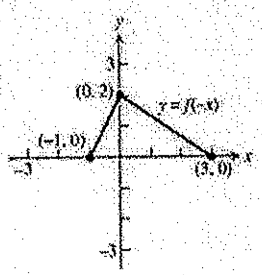
Problem 3

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$

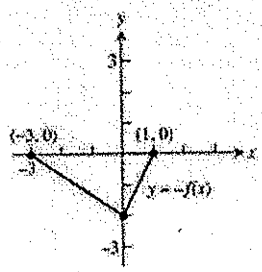
$\Rightarrow (1 - \cos x) = \frac{\sin^2 x}{1+\cos x} \quad \Rightarrow \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

Problem 4

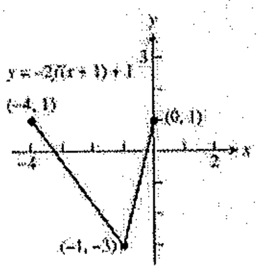
(a) The given graph is reflected about the y-axis.



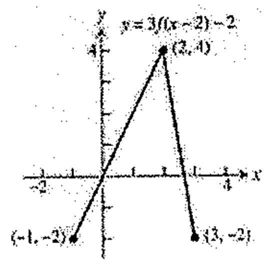
(b) The given graph is reflected about the x-axis.



(c) The given graph is shifted left 1 unit, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted upward 1 unit.

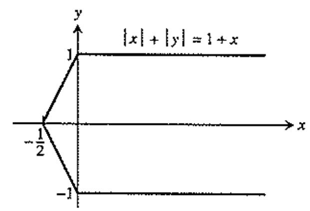


(d) The given graph is shifted right 2 units, stretched vertically by a factor of 3, and then shifted downward 2 units.



Extra

(a) For (x, y) in the 1st quadrant, $|x| + |y| = 1 + x$ $\Leftrightarrow x + y = 1 + x \Leftrightarrow y = 1$. For (x, y) in the 2nd quadrant, $|x| + |y| = x + 1 \Leftrightarrow -x + y = x + 1$ $\Leftrightarrow y = 2x + 1$. In the 3rd quadrant, $|x| + |y| = x + 1$ $\Leftrightarrow -x - y = x + 1 \Leftrightarrow y = -2x - 1$. In the 4th quadrant, $|x| + |y| = x + 1 \Leftrightarrow x + (-y) = x + 1$ $\Leftrightarrow y = -1$. The graph is given at the right.



(b) We use reasoning similar to Exercise 7.

(1) 1st quadrant: $y + |y| = x + |x|$
$\Leftrightarrow 2y = 2x \Leftrightarrow y = x$.

(2) 2nd quadrant: $y + |y| = x + |x|$
$\Leftrightarrow 2y = x + (-x) = 0 \Leftrightarrow y = 0$.

(3) 3rd quadrant: $y + |y| = x + |x|$
$\Leftrightarrow y + (-y) = x + (-x) \Leftrightarrow 0 = 0$
$\Rightarrow$ <u>all</u> points in the 3rd quadrant satisfy the equation.

(4) 4th quadrant: $y + |y| = x + |x|$
$\Leftrightarrow y + (-y) = 2x \Leftrightarrow 0 = x$. Combining these results we have the graph given at the right: