

MAS115 Calculus I 2007-2008

Problem sheet for exercise class 1

- **Make sure you attend the exercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

(*) Problem 1: Determine the set of all real numbers x (i.e. $x \in \mathbb{R}$) that satisfy

$$x^2 - 3x - 4 < 0$$

- (a) by direct computation, and
 - (b) by plotting the graph of $y = x^2 - 3x - 4$.
- Hint: compute the zeros of $x^2 - 3x - 4$.

Problem 2: Determine the set of all real numbers x (i.e. $x \in \mathbb{R}$) that satisfy

$$|2x - 1| + |4x + 1| < 3$$

- (a) by direct computation, and
- (b) by plotting the graph.

Problem 3: Determine the set of all real numbers x (i.e. $x \in \mathbb{R}$) that satisfy

$$\sqrt{1 - x^2} \leq -x$$

- (a) by direct computation, and
- (b) by plotting the graphs of $y = -x$ and $y = \sqrt{1 - x^2}$.

Problem 4: Prove that for all positive real numbers x and y (i.e. $x, y \in \mathbb{R}^+$),

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy}$$

- (a) by direct proof, and
- (b) by using the geometric-arithmetic inequality.

Extra: Prove that for all real numbers x and y (i.e. $x, y \in \mathbb{R}$)

$$||x| - |y|| = |x + y| + |x - y| - |x| - |y|.$$

(*) Problem 1

$$x^2 - 3x - 4 < 0$$

-1-

Hint: $x^2 - 3x - 4 = 0$

$$\Leftrightarrow x_{1/2} = \frac{3 \pm \sqrt{9+16}}{2}$$

$$x_1 = -1, x_2 = 4$$

$$x^2 - 3x - 4 = (x - x_1)(x - x_2)$$

$$= (x+1)(x-4)$$

$$x^2 - 3x - 4 < 0 \Leftrightarrow (x+1)(x-4) < 0$$

Therefore (a) $x+1 > 0$ and $x-4 < 0$

or (b) $x+1 < 0$ and $x-4 > 0$

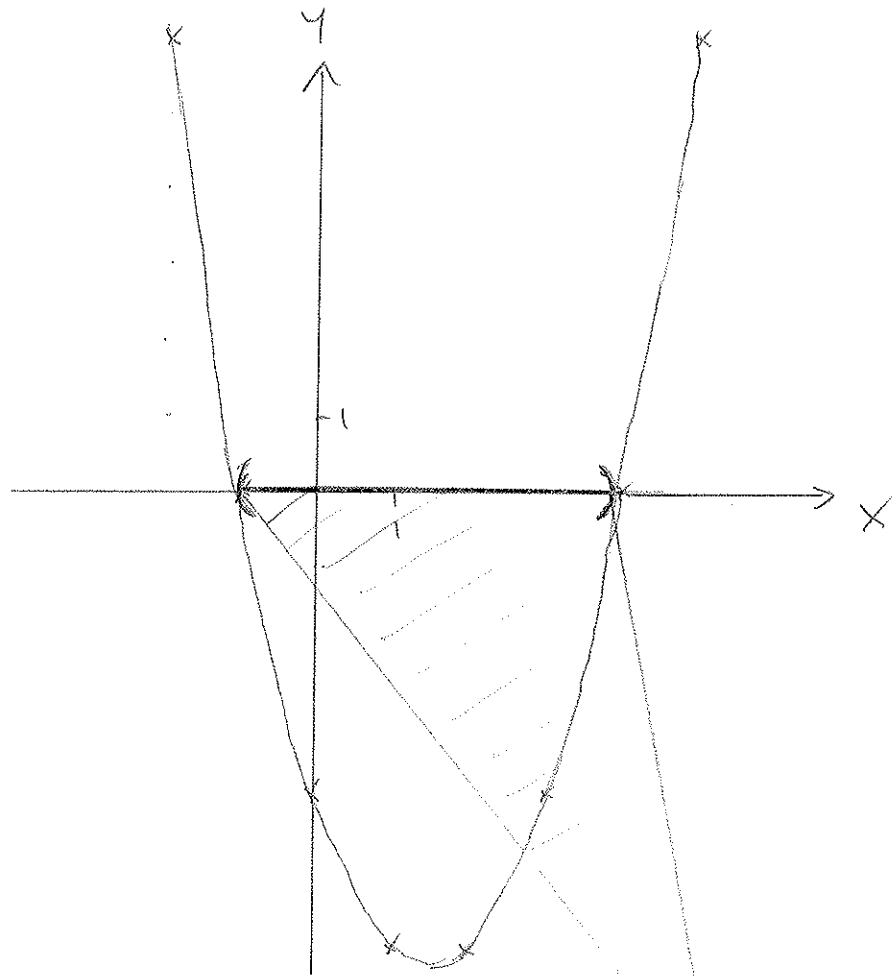
(a) gives $x > -1$ and $x < 4 \Rightarrow x \in (-1, 4)$

(b) gives $x < -1$ and $x > 4 \Rightarrow x \in \emptyset$

Solution: $x \in (-1, 4)$

Graph of $y = x^2 - 3x - 4 = (x-4)(x+1)$

x	y
-3	14
-2	6
-1	0
0	-4
1	-6
2	-6
3	-4
4	0
5	6
6	14



$$x^2 - 3x - 4 < 0 \Leftrightarrow -1 < x < 4$$

Problem 2

$$|2x-1| + |4x+1| < 3$$

(a) $2x-1 < 0$ and $4x+1 < 0$

$$\Leftrightarrow x < \frac{1}{2} \quad \text{and} \quad x < -\frac{1}{4}$$

i.e. $x \in (-\infty, -\frac{1}{4})$

Then $|2x-1| + |4x+1|$

$$= -(2x-1) - (4x+1) = -6x$$

$$-6x < 3 \Leftrightarrow x > -\frac{1}{2}$$

so $x \in (-\infty, -\frac{1}{4}) \cap (-\frac{1}{2}, \infty) = \underline{\underline{(-\frac{1}{2}, -\frac{1}{4}]}}$

(b) $2x-1 < 0$ and $4x+1 \geq 0$

$$\Leftrightarrow x < \frac{1}{2} \quad \text{and} \quad x \geq -\frac{1}{4}$$

i.e. $x \in [-\frac{1}{4}, \frac{1}{2})$

Then $|2x-1| + |4x+1|$

$$= -(2x-1) + (4x+1) = 2x+2$$

$$2x+2 \leq 3 \Leftrightarrow x \leq \frac{1}{2}$$

$$\text{so } x \in \left[-\frac{1}{4}, \frac{1}{2}\right) \cap \left(-\infty, \frac{1}{2}\right) = \underline{\underline{\left[-\frac{1}{4}, \frac{1}{2}\right)}}$$

(c) $2x-1 \geq 0$ and $4x+1 < 0$

$$\Leftrightarrow x \geq \frac{1}{2} \quad \text{and} \quad x < -\frac{1}{4}$$

this is not possible for any $x \in \mathbb{R}$; $x \in \emptyset$

(d) $2x-1 \geq 0$ and $4x+1 \geq 0$

$$\Leftrightarrow x \geq \frac{1}{2} \quad \text{and} \quad x \geq -\frac{1}{4}$$

$$\text{ie. } x \in \left[\frac{1}{2}, \infty\right)$$

Then $|2x-1| + |4x+1|$

$$= (2x-1) + (4x+1) = 6x$$

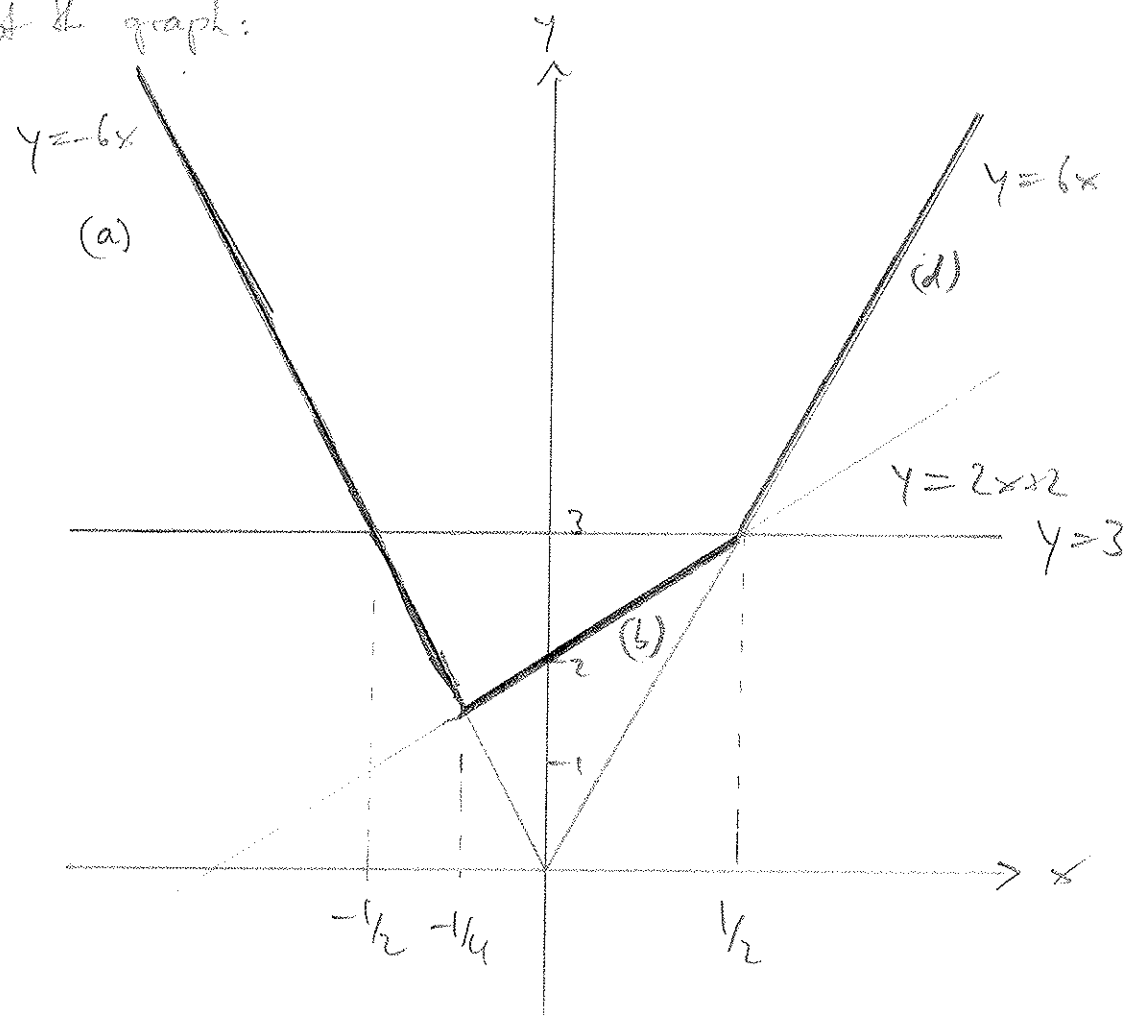
$$6x < 3 \Leftrightarrow x < \frac{1}{2}$$

$$\text{so } x \in \left[\frac{1}{2}, \infty\right) \cap \left(-\infty, \frac{1}{2}\right) = \underline{\underline{\emptyset}}$$

Together,

$$\underline{\underline{x \in \left(-\frac{1}{2}, -\frac{1}{4}\right] \cup \left(-\frac{1}{4}, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)}}$$

Plot the graph:



Problem 3

$$\sqrt{1-x^2} \leq -x$$

Observe

- lhs is only defined for $x \in [-1, 1]$
- lhs is non-negative, therefore $x \leq 0$

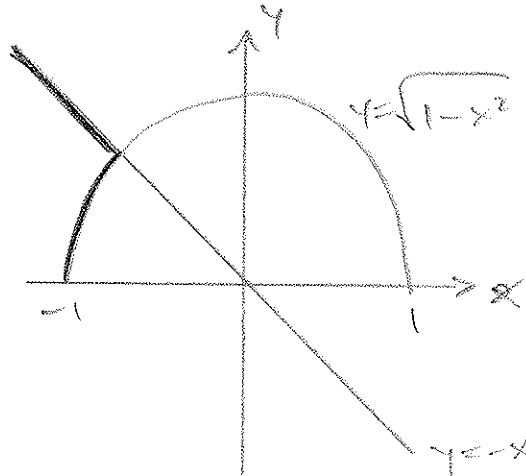
squaring both sides gives

$$1-x^2 \leq x^2$$

$$\Leftrightarrow \frac{1}{2} \leq x^2$$

$$\Leftrightarrow -x \leq -\frac{\sqrt{2}}{2} \text{ or } x \geq +\frac{\sqrt{2}}{2}$$

but $x \in [-1, 0]$, so $x \in [-1, -\frac{\sqrt{2}}{2}]$



Thus (*) is equivalent to

$$0 \leq (x-y)^2$$

which is true for all $x, y \geq 0$

(b) Using $\sqrt{xy} \leq \frac{1}{2}(x+y)$

Rewrite $\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy} \quad | \cdot \frac{1}{2}(\frac{1}{x} + \frac{1}{y})$

$$\Leftrightarrow 1 \leq \sqrt{xy} \frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right) \quad | \cdot \frac{1}{\sqrt{xy}}$$

$$\Leftrightarrow \frac{1}{\sqrt{xy}} \leq \frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right)$$

$$\Leftrightarrow \sqrt{\frac{1}{x}} \sqrt{\frac{1}{y}} \leq \frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right)$$

this is $\sqrt{xy} \leq \frac{1}{2}(x+y)$ with x, y

replaced by $\frac{1}{x}$ and $\frac{1}{y}$

Extra lots of different cases

$$x \geq y \geq 0, \quad x \geq 0 \geq y, \quad 0 \geq x \geq y$$

$$x \leq y \leq 0, \quad x \leq 0 \leq y, \quad 0 \leq x \leq y$$

e.g. for $x \geq 0 \geq y$

the lhs gives

$$||x| - |y|| = |x + y|$$

and the rhs gives

$$|x + y| + |x - y| = |x| - |y|$$

$$= |x + y| + x - y - x + y = |x + y|$$

which are equal.