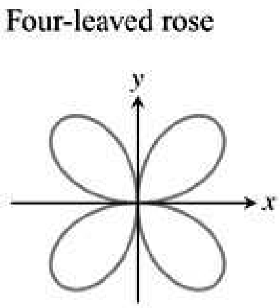
MAS115 Calculus I 2007-2008

Problem sheet for exercise class 10



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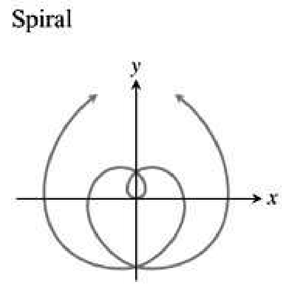

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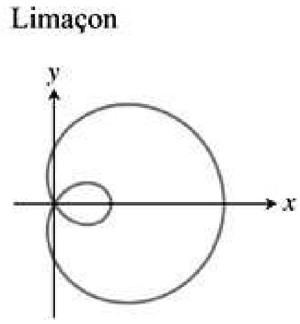

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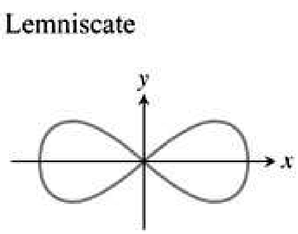
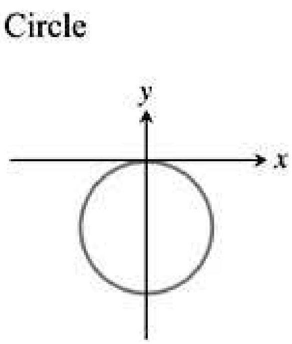
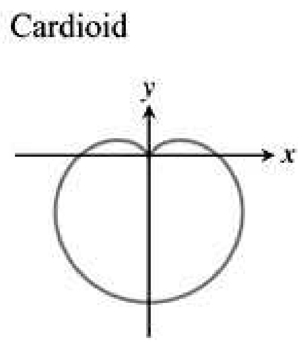
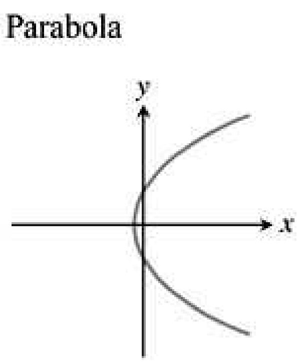
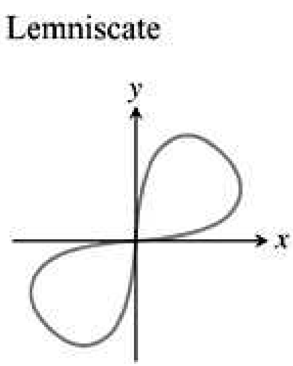










Problem 1: Match each of the eight graphs with one of the following equations.

a. $r = \cos 2\theta$, **b.** $r\cos\theta = 1$, **c.** $r = \dfrac{6}{1-2\cos\theta}$,

d. $r = \sin 2\theta$, **e.** $r = \theta$, **f.** $r^2 = \cos 2\theta$,

g. $r = 1+\cos\theta$, **h.** $r = 1-\sin\theta$, **i.** $r = \dfrac{2}{1-\cos\theta}$,

j. $r^2 = \sin 2\theta$, **k.** $r = -\sin\theta$, **l.** $r = 2\cos\theta + 1$.

Problem 2: Show that the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the polar equation

$$r = \frac{k}{1 + e\cos\theta}$$

into the Cartesian equation

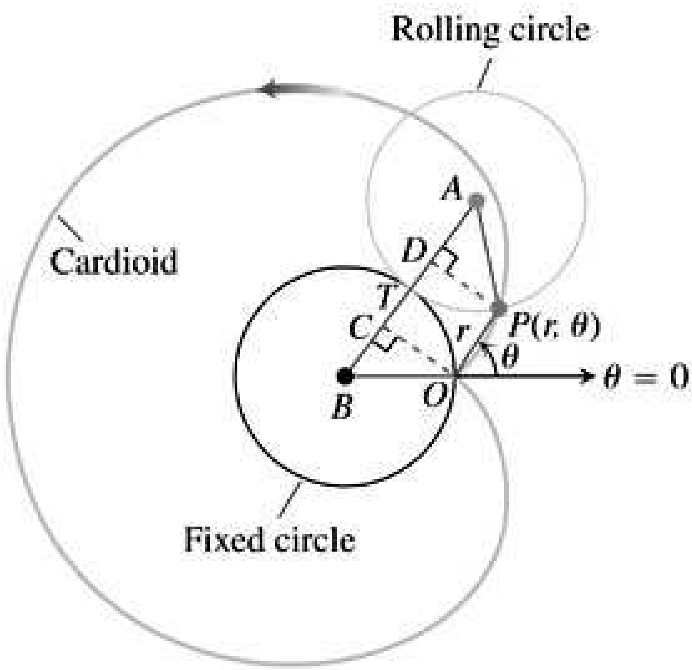
$$(1 - e^2)x^2 + y^2 + 2kex - k^2 = 0 \ .$$

Problem 3: Find polar equations for the following four circles. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

a. $x^2 + y^2 + 5y = 0$, **b.** $x^2 + y^2 - 2y = 0$,

c. $x^2 + y^2 - 3x = 0$, **d.** $x^2 + y^2 + 4x = 0$.

Extra: Show that if you roll a circle of radius a about another circle of radius a in the polar coordinate plane, the original point of contact P will trace a cardioid. (Hint: start by showing that $\angle OBC$ and $\angle PAD$ are equal to each other.)



Thomas Prellberg, December 2007

Problem 1:

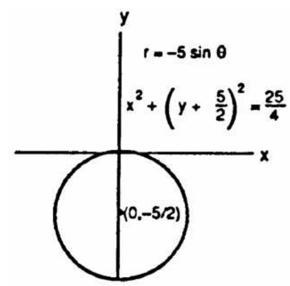
rose: d; spiral: e; limaçon: l; lemniscate: f; circle: k; cardioid: h; parabola: i; lemniscate (diagonal): j.

Problem 2:

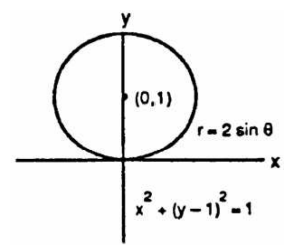
$r = \frac{k}{1 + e\cos\theta} \Rightarrow r + er\cos\theta = k \Rightarrow \sqrt{x^2 + y^2} + ex = k \Rightarrow \sqrt{x^2 + y^2} = k - ex \Rightarrow x^2 + y^2 = k^2 - 2kex + e^2x^2 \Rightarrow x^2 - e^2x^2 + y^2 + 2kex - k^2 = 0 \Rightarrow (1 - e^2)\,x^2 + y^2 + 2kex - k^2 = 0$

Problem 3:

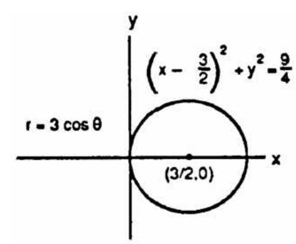
a.
$x^2 + y^2 + 5y = 0 \Rightarrow x^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{4} \Rightarrow C = \left(0, -\frac{5}{2}\right)$ and $a = \frac{5}{2}$; $r^2 + 5r\sin\theta = 0 \Rightarrow r = -5\sin\theta$



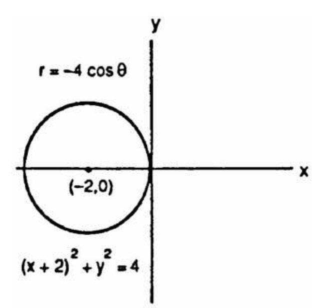
b.
$x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y - 1)^2 = 1 \Rightarrow C = (0, 1)$ and $a = 1$; $r^2 - 2r\sin\theta = 0 \Rightarrow r = 2\sin\theta$



c.
$x^2 + y^2 - 3x = 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4} \Rightarrow C = \left(\frac{3}{2}, 0\right)$ and $a = \frac{3}{2}$; $r^2 - 3r\cos\theta = 0 \Rightarrow r = 3\cos\theta$



d.
$x^2 + y^2 + 4x = 0 \Rightarrow (x + 2)^2 + y^2 = 4 \Rightarrow C = (-2, 0)$ and $a = 2$; $r^2 + 4r\cos\theta = 0 \Rightarrow r = -4\cos\theta$



Problem 4:

Arc PT = Arc TO since each is the same distance rolled. Now Arc PT = a($\angle$TAP) and Arc TO = a($\angle$TBO) $\Rightarrow$ $\angle$TAP = $\angle$TBO. Since AP = a = BO we have that ΔADP is congruent to ΔBCO $\Rightarrow$ CO = DP $\Rightarrow$ OP is parallel to AB $\Rightarrow$ $\angle$TBO = $\angle$TAP = θ. Then OPDC is a square $\Rightarrow$ r = CD = AB − AD − CB = AB − 2CB $\Rightarrow$ $r = 2a - 2a\cos\theta = 2a(1 - \cos\theta)$, which is the polar equation of a cardioid.