4. Geometric Theories

4.1 Quillen's Approach to Cobordism Theories

We shall be defining various bordism theories $h_*(-)$ and from time to time will need to examine the dual cobordism $h^*(-)$. As we shall only need a picture of $h^*(X)$ when X is a manifold (usually $S^{2n+1}/_{Z/p}$), Quillen's approach to cobordism (33), which uses Poincaré duality, will give an appropriate geometric description of cobordism classes. As an example we sketch his definition of $U^*(X)$. For the definition it is necessary to assume X is a manifold. (However there is no great loss of generality as we allow non-compact manifolds and any finite CW complex X can be embedded in some R^n with regular neighbourhood such a manifold.)

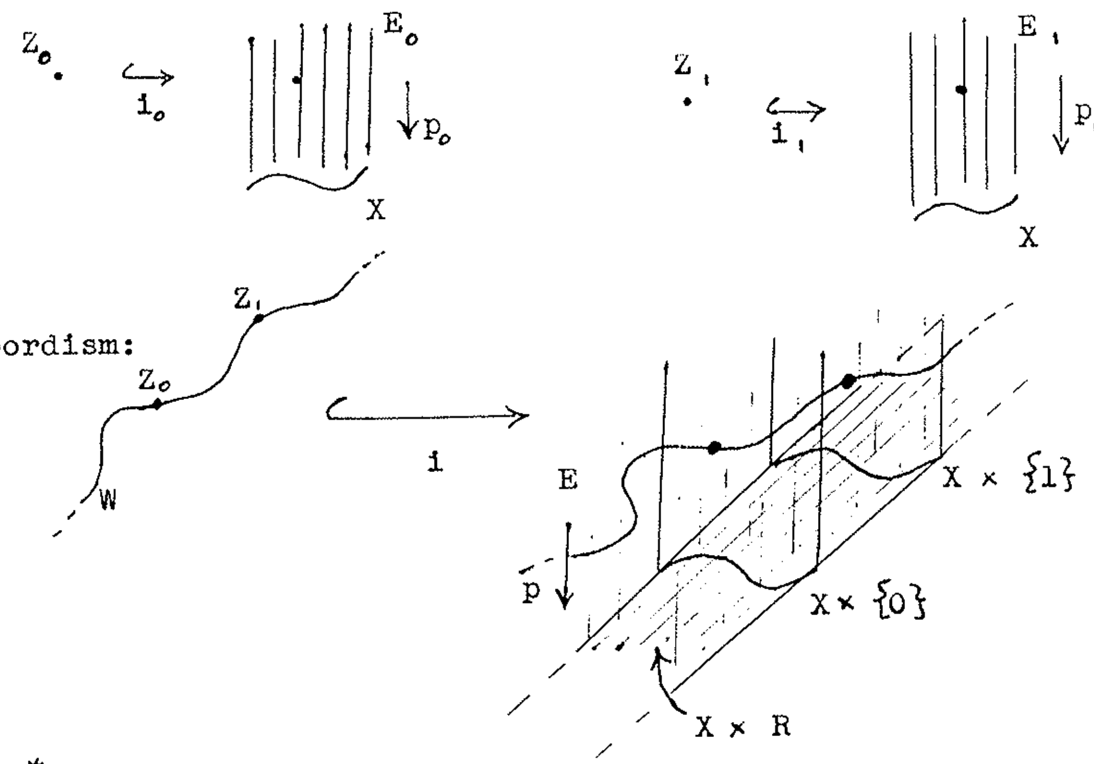
<u>Definition 4.1.1 (Quillen (33))</u> Let $f:Z^{n-q} \to X^n$ be a map of manifolds (without boundary). A <u>U-orientation</u> for f is an equivalence class of factorisations of f:

$$Z \overset{i}{\rightarrowtail} E \xrightarrow{p} X$$

where $p:E \to X$ is a complex vector bundle over X (or, if q is odd E is (complex vector bundle)$\times R$) and i is an embedding with a complex structure on its normal bundle ν_i. Two such factorisations i, i' through E, E' are <u>equivalent</u> if there is a bundle E'' containing E and E' as summands, such that within E'' i is isotopic to i' compatibly with the normal structure. The <u>dimension</u> of this oriented map is q.

<u>Definition 4.1.2 (Quillen (33))</u> Two proper U-oriented maps $f_0:Z_0 \to X$, $f_1:Z_1 \to X$ are said to be <u>cobordant</u> if there is a proper U-oriented map $F:W \to X\times R$ such that the maps $\varepsilon_i:X \to X\times R$ ($\varepsilon_i(x) = (x,i)$ $i = 0,1$) are transverse to F and the pull-back of F by ε_i gives f_i. ($i = 0,1$)

Picture of a cobordism between U-oriented maps $f_0 : Z_0 \longrightarrow X$, $f_1 : Z_1 \longrightarrow X$:-



Let $U^*(-)$ be the cohomology theory defined by the spectrum $\underset{\sim}{MU}$.

Proposition 4.1.3 (Quillen (33)) $U^q(X)$ is naturally isomorphic to the set of cobordism classes of proper U-oriented maps of dimension q. Also, if $Y \hookrightarrow X$ is a deformation retract of a regular neighbourhood, $U^q(X, X \setminus Y)$ is canonically isomorphic to the set of cobordism classes of proper U-oriented maps with image contained in Y.

Proof: Quillen (33). It is just the "Thom construction". (Note that we need proper maps (i.e. inverse image of any compact set compact) in order to use the Thom transversality technique.) ▮

Proposition 4.1.4 $U^*(-)$ is the dual cobordism theory to $U_*(-)$. (i.e. on finite complexes they may be defined by the same spectrum) Recall $U_*(-)$ is the bordism group of compact manifolds with complex structure on their stable normal bundle.

<u>Proof</u> This is trivial if we define $U_*(-)$ and $U^*(-)$ as the homology and cohomology theories corresponding to the spectrum $\underset{\sim}{MU}$. However we shall be dealing with geometrically defined bordism and cobordism theories without investigating their spectra, so we outline a proof regarding $U^*(-)$ as cobordism of proper U-oriented maps:-

We have to prove $U^*(-)$ and $U_*(-)$ are Spanier-Whitehead dual (see Whitehead (42)) i.e. for X embedded in S^n as the deformation retract of a regular neighbourhood we have to show there is a natural isomorphism $\phi : U^q(S^n, S^n \setminus X) \cong U_{n-q}(X)$.

<u>Definition of ϕ</u>

Let $x \in U^q(S^n, S^n \setminus X)$

Represent x by the U-oriented map:
$$\begin{array}{c} M \xhookrightarrow{i} E \\ \quad\;\; \downarrow p \\ \quad\;\; S^n \end{array} \quad \text{(with image } \subset X)$$

E is always a summand of a trivial complex vector bundle so without loss of generality we may take E trivial.

ν_i now gives a U-structure to M, (i.e. a complex structure on the stable normal bundle of M), and pi proper $\Rightarrow$ M compact. Thus $M \xrightarrow{pi} X$ is an element of $U_{n-q}(X)$. It is straightforward to check ϕ is well-defined.

<u>Definition of ϕ^{-1}</u>

Let $y \in U_{n-q}(X)$

Represent y by $M \xrightarrow{f} X$ ~~and homotop f to miss a regular neighbourhood of $S^n \setminus X$~~. (Now note that since every compact subset of S^n is closed, M compact $\Rightarrow$ f proper.)

Now take any embedding $M \overset{i}{\hookrightarrow} C^N$ for some large N and consider the factorisation of f:-

$$\begin{array}{ccc} M & \xhookrightarrow[f \times i]{} & S^n \times C^N \\ & & \downarrow \pi_1 \\ & & S^n \end{array}$$

Since M is a U-manifold we have a complex structure on $\nu_{f\times i}$. Standard theorems about embeddings of M in C^N being isotopic for large N then show that $\phi^{-1}y$ is well-defined and that ϕ^{-1} is inverse to ϕ. ∎

4.2 "Euler Classes" for Z/p-bundles

This section is intended mainly to give some motivation for the construction of certain bordism and cobordism theories in the following sections.

Recall definition 3.3.1: a representative Z/p-theory $h^*(-)$ is a connective commutative ring theory with $h^0(pt.)\cong Z/p$, together with classes $\begin{pmatrix}\alpha_h\\ \beta_h\end{pmatrix}\in\begin{pmatrix}h^1(BZ/p)\\ h^2(BZ/p)\end{pmatrix}$ which map to $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\in\begin{pmatrix}H^1(BZ/p;Z/p)\\ H^2(BZ/p;Z/p)\end{pmatrix}$, and which has $\beta_h\in$ image $i^*:h^2(BS^1)\rightarrowtail h^2(BZ/p)$

For such a theory $h^*(X)\xrightarrow{\mu}H^*(X;Z/p)$ is epimorphic for all X (Corollary 3.1.3 (Rourke)) and also $m:Z/p\times Z/p\longrightarrow Z/p$ gives a Z/p-formal group structure to $h^*(BZ/p)$ (Definition 3.4.1

We may therefore expect the universal Z/p-formal group to have a geometric realisation as "the universal representative Z/p-theory". So we start by examining the geometry of BZ/p:-

BZ/p may be regarded as the limit:

$$S^{2n-1}/Z/p\longrightarrow S^{2n+1}/Z/p\longrightarrow S^{2n+3}/Z/p\longrightarrow\cdots$$

(where the action of Z/p on S^{2n-1} is given by taking the complex S^1-action on $S^{2n-1}\hookrightarrow C^n$ and using our standard $Z/p\xrightarrow[i]{}S^1$)

Consider the embedded U-submanifold: $S^{2n-1}/Z/p\xhookrightarrow[i_n]{}S^{2n+1}/Z/p$

This represents a class $\beta_{(n)}\in H_{2n-1}(S^{2n+1}/Z/p;Z/p)$ and thus, by Poincaré duality, a class $\beta^{(n)}\in H^2(S^{2n+1}/Z/p;Z/p)$.

$i_n^*\beta^{(n)}=\beta^{(n-1)}$ and $\varprojlim\beta^{(n)}=\beta\in H^2(BZ/p;Z/p)$ (see e.g. Rourke (34))

Similarly consider the embedded "U-submanifold with Z/p-singularity": $S^{2n-1} * Z/p \big/ Z/p \hookrightarrow S^{2n+1}/Z/p$ (Z/p has the join action on $S^{2n-1} * Z/p$)

This represents a class $\alpha_{(n)} \in H_{2n}(S^{2n+1}/Z/p; Z/p)$ and thus, by Poincaré duality, a class $\alpha^{(n)} \in H^1(S^{2n+1}/Z/p; Z/p)$.

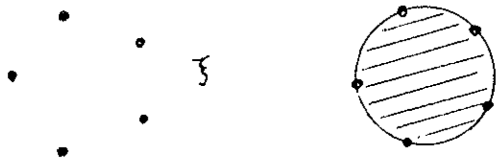
$i_n^* \alpha^{(n)} = \alpha^{(n-1)}$ and $\varprojlim \alpha^{(n)} = \alpha \in H^1(BZ/p; Z/p)$ (see e.g. Rourke (34); note that $S^{2n-1} * Z/p \big/ Z/p \cong D^{2n}/\sim$ where $\sim$ represents the identification $S^{2n-1} \twoheadrightarrow S^{2n-1}/Z/p$.)

We shall consider the bordism theory ${}^{(1)}V_*(-)$ of "U-manifolds with Z/p-singularity". The above definitions then give $\beta_{(n)} \in {}^{(1)}V_{2n-1}(S^{2n+1}/Z/p)$ and $\alpha_{(n)} \in {}^{(1)}V_{2n}(S^{2n+1}/Z/p)$. Poincaré duality gives elements $\beta^{(n)} \in {}^{(1)}V^2(S^{2n+1}/Z/p)$ and $\alpha^{(n)} \in {}^{(1)}V^1(S^{2n+1}/Z/p)$ and using the Quillen picture for ${}^{(1)}V^*(-)$ it will be clear that $\alpha^{(n)}, \beta^{(n)}$ give well-defined $\alpha_V \in \varprojlim_n {}^{(1)}V^1(S^{2n+1}/Z/p)$ and $\beta_V \in \varprojlim_n {}^{(1)}V^2(S^{2n+1}/Z/p)$.

Thus in the cobordism theory ${}^{(1)}V^*(-)$ (formal definition later, in 4.3) each Z/p-bundle has natural characteristic classes α_V, β_V, which we now show are closely analogous to the Euler class e_N of a $Z/2$-bundle.

In the $Z/2$-case $RP^{n+1} \simeq$ Thom space of the line bundle over RP^n (= mapping cone of $Z/2$-bundle over RP^n). $e_N^{(n)}$ is represented by the embedded submanifold $RP^n \hookrightarrow RP^{n+1}$ and this choice gives a well-defined element e_N of $\varprojlim N^1(RP^n)$ as $i^* e_N^{(n)} = e_N^{(n-1)}$.

In the Z/p case we have a more complicated construction on the Z/p-bundle ζ over $S^{2n-1}/Z/p$ to get $S^{2n+1}/Z/p$ ($\cong (S^{2n-1} * S^1)/Z/p$):



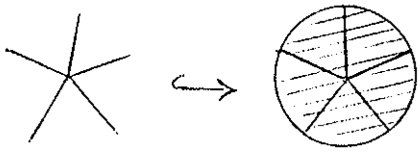
$D(\xi) = D^2$-bundle associated to ξ

Each fibre of $D(\xi)$ has boundary S^1 with a Z/p-action on it: we glue all these S^1's together to a single $S^1/_{Z/p}$.

<u>Definition 4.2.1</u> We may apply this construction to any Z/p-bundle η. We call the resulting space the <u>quasi-Thom space</u> $q(\eta)$ of η. Thus $q(\xi) \simeq S^{2n+1}/_{Z/p}$.

<u>Definition 4.2.2</u> The mapping cone of a Z/p-bundle η is called the <u>Thom space</u> $t(\eta)$ of η.

$t(\eta) \hookrightarrow q(\eta)$



(with identifications on boundaries)

Let $b(\eta)$ denote the zero section of $q(\eta)$.

Then for the Z/p-bundle ξ over $S^{2n-1}/_{Z/p}$ we may represent:

$\alpha_V \in {}^{(1)}V^1(S^{2n+1}/_{Z/p})$ as the class Poincaré dual to the embedded "U-manifold with Z/p-singularity" $t(\xi) \hookrightarrow q(\xi)$

$\beta_V \in {}^{(1)}V^2(S^{2n+1}/_{Z/p})$ as the class Poincaré dual to the embedded U-manifold $b(\xi) \hookrightarrow q(\xi)$

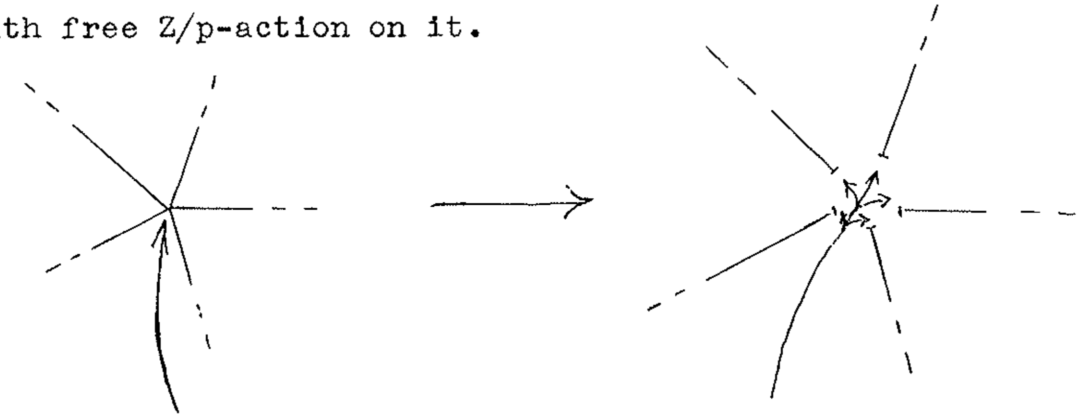
(Once again, formal explanations later)

This choice of α_V, β_V will give well-defined elements $\in \varprojlim_n {}^{(1)}V^*(S^{2n+1}/_{Z/p})$ since they pull back correctly under $S^{2n-1}/_{Z/p} \hookrightarrow S^{2n+1}/_{Z/p}$.

<u>Remark 4.2.3</u> By its definition this β_V represents the U^*--Euler class of the complex line bundle associated to ξ, so $\beta_V \in$ image $i^*: {}^{(1)}V^*(BS^1) \to {}^{(1)}V^*(BZ/p)$ (since it is in the image: $U^*(BS^1) \to {}^{(1)}V^*(BS^1) \to {}^{(1)}V^*(BZ/p)$.)

4.3 U-manifolds with Z/p-singularities: ${}^{(1)}V_*(-)$

The object of this section is to give a formal definition of "U-manifolds with Z/p-singularities"; the easiest approach is to cut along the Z/p-singularity set, leaving a boundary with free Z/p-action on it.



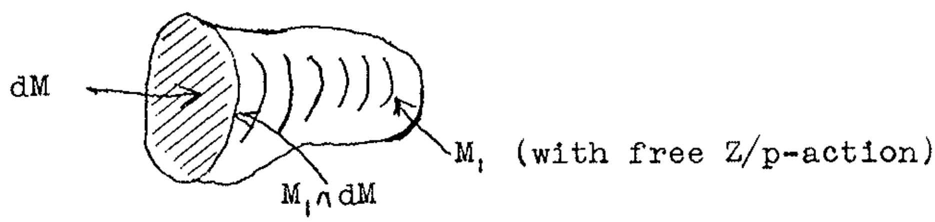
singularity set

boundary with free Z/p-action on it

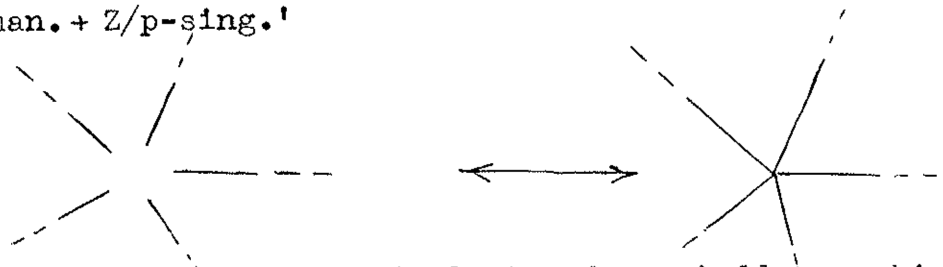
Definition 4.3.1 (M, M_1, t) is a U-manifold with Z/p-singular-ity (= a ${}^{(1)}V$-manifold) if M is a U-manifold with boundary M_1 and a free action t of Z/p on M_1, preserving the U-structure on M_1.

(A U-manifold M is a manifold with a given complex structure on its stable tangent bundle τM; as M_1 is the boundary of M, it has a collar in M and so $\tau M|_{M_1} = \tau M_1 \oplus$(trivial real line bundle corres-ponding to inward normal) and this induces a U-structure on M_1)

(M, dM, M_1, t) is a U-man.+ Z/p-sing. with boundary (= a ${}^{(1)}V$-man. with boundary) if M is a U-man. with boundary $M_1 \cup dM$, such that dM is a U-man. with boundary $M_1 \cap dM$, and t is a free action of Z/p on M_1 preserving the U-structures on M_1 and $M_1 \cap dM$. (so that dM is a ${}^{(1)}V$-man.)



Remark 4.3.2 Given a ${}^{(1)}V$-man. (M, M_1, t), if we quotient by the Z/p-action t we get a 'pseudo-manifold' where each point has link S^{n-1} or $S^{n-2} * Z/p$; the singularity stratum (set of points with link $S^{n-2} * Z/p$) is a codimension one submanifold having normal bundle with 'fibre' $Z/p * pt.$ and structure group Z/p. Thus 4.3.1 does indeed capture the notion of 'U-man. + Z/p-sing.'



The two pictures are equivalent and we shall use whichever is most convenient for each situation.

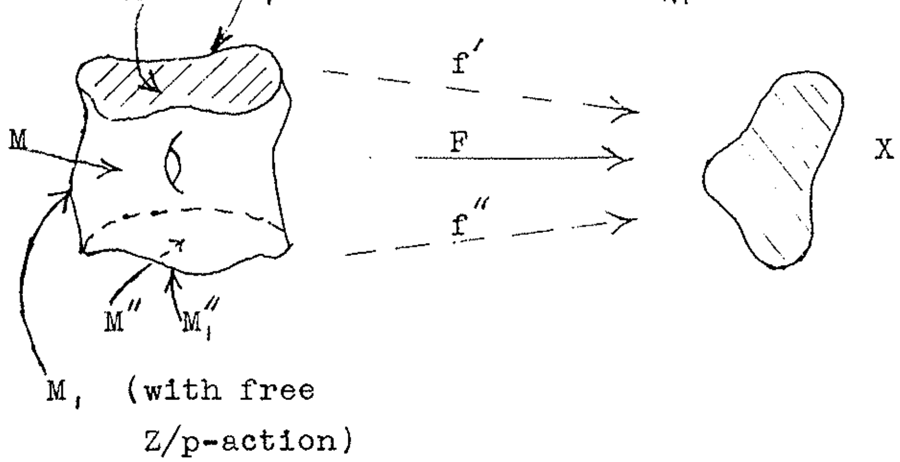
Remark 4.3.3 Our definition is just that of Sullivan (40), Baas (3), Stone (38) etc. for manifolds with singularities, except that Sullivan and Baas deal only with the case where M_1 is a trivial Z/p-bundle i.e. when the normal bundle to the singularity stratum is trivial.

Remark 4.3.4 It is important for transversality arguments later to observe that being the boundary of M, M_1 has a collar in M, i.e. a regular neigbourhood $\cong M_1 \times I$. After quot--ienting by the Z/p-action this collar becomes a regular neighbourhood of the singularity set $M_1/_{Z/p}$, indexed by I.

Definition 4.3.5 ${}^{(1)}V$-manifolds M', M'' are said to be bordant if there is a ${}^{(1)}V$-manifold with boundary (M, dM) such that $dM \cong M' \amalg (-M'')$ ("-" denotes reverse U-structure). Under disjoint union as sum the bordism classes form a group ${}^{(1)}V_*$.

Given any space X the bordism group ${}^{(1)}V_*(X)$ is defined to be the group of equivalence classes of maps $f: M \longrightarrow X$ (where M is a compact ${}^{(1)}V$-manifold and $f|_{M_1}$ is equivariant with respect to the given Z/p-action on M_1 and the trivial Z/p-action on X). The equivalence relation is that:

$$(f': M' \longrightarrow X) \sim (f'': M'' \longrightarrow X) \Longleftrightarrow \exists M \text{ such that } \partial M = M' \amalg (-M'')$$
$$\& \ F: M \longrightarrow X \text{ such that } F|_{M'} = f'$$
$$\& \ F|_{M''} = f'' \ (F \text{ equivariant on } M_1)$$



Once again we have two equivalent 'pictures' of the bordism group:-

(1) Manifolds with boundary and free Z/p-action on that boundary (and equivariant maps f).

(2) Manifolds with Z/p-singularities (and maps f).

<u>Theorem 4.3.6</u> ${}^{(1)}V_*(-)$ is a generalised homology theory.

<u>Proof</u> The proof that ${}^{(1)}V_*(-)$ satisfies the Eilenberg-Steenrod axioms follows the lines of the standard geometric proofs that $N_*(-)$ is a homology theory. (see e.g. Bröcker and tom Dieck (6)); rather than present a full proof we indicate the (Chapt.II) changes necessary to Bröcker & tom Dieck's proof for the most difficult axiom, Mayer-Vietoris:-

Let $X = X_0 \cup X_1$ be a normal space with X_0, X_1 open subspaces. We have to construct a natural exact sequence:-

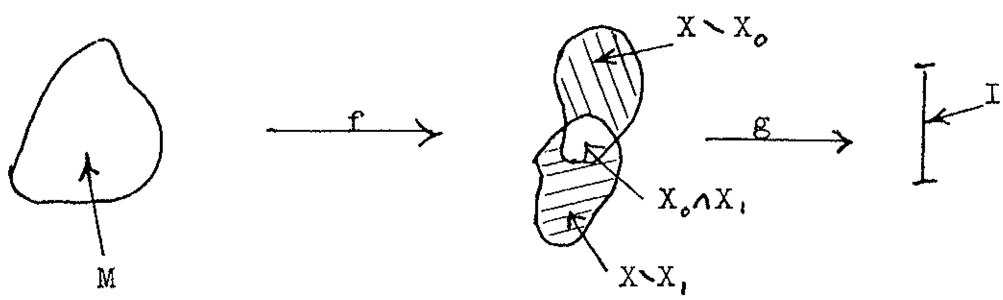
$$\ldots \to {}^{(1)}V_n(X_0 \cap X_1) \to {}^{(1)}V_n(X_0) \oplus {}^{(1)}V_n(X_1) \to {}^{(1)}V_n(X) \xrightarrow{\delta}{}^{(1)}V_{n-1}(X_0 \cap X_1) \to \ldots$$

Each map here has an obvious geometric definition, except for δ.

Definition of δ

Let $x \in {}^{(1)}V_n(X)$ be represented by the pair (M,f), where M is a ${}^{(1)}V$-manifold and $f: M \to X$.

By Urysohn's lemma there is a continuous map $g: X \to I$ such that $g^{-1}(0) \supset (X \setminus X_0)$ and $g^{-1}(1) \supset (X \setminus X_1)$. Let $h = gf: M \to I$



What is now needed is a transversality theorem to give arbitrarily close to h an h', such that h' is transverse to the submanifold $\{\frac{1}{2}\} \hookrightarrow I$. (Regard M in this instance as a manifold with singularities: 'transverse' for such a manifold means 'simultaneously transverse on all strata'. For details see e.g. Stone (38))

The method of constructing such an h' is:

(1) On the singularity stratum $M_1/_{Z/p}$ we homotop h to h_1, transverse to $\{\frac{1}{2}\}$. (using Sard's theorem)

(2) Using the regular neighbourhood of the singularity stratum given in 4.3.4 we extend h_1 to a map h_2 defined on the whole of M and homotopic to h. (using the homotopy of (1) to make this extension.)

(3) Keeping h_2 fixed on the singularity stratum we homotop h_2 to h' transverse to $\{\frac{1}{2}\}$ away from that stratum. (using Sard's theorem)

δM is now defined to be $h'^{-1}\{(\frac{1}{2})\}$, and the transversality of h' ensures that $\delta M \in {}^{(1)}V_{n-1}(X_0 \cap X_1)$. It is well-defined as a bordism class by the standard arguments of Bröcker and tom Dieck (6), (Chapt. II) with 'manifold' replaced by '${}^{(1)}V$-manifold', and the Mayer-Vietoris sequence may be proved exact for ${}^{(1)}V_*(-)$ by their (geometric) method.

By its definition δ is natural, and ${}^{(1)}V_*(-)$ clearly satisfies the homotopy axiom (by regarding a homotopy as a bordism), so all the Eilenberg-Steenrod axioms are satisfied. ∎

We now proceed to define the dual cobordism theory ${}^{(1)}V^*(-)$ in an analogous way to Quillen's definition of $U^*(-)$ (4.1). ${}^{(1)}V^*(X)$ will be defined only for X a manifold, but as explained in 4.1 this is sufficient to define it for X a finite CW complex X. We may use either 'picture' of ${}^{(1)}V$-manifold, but choose that of 'U-manifold with Z/p-singularity' rather than the 'cut-open' picture, as it seems the most appropriate form for the class $\alpha_V \in {}^{(1)}V^1(BZ/p)$:-

<u>Definition 4.3.7</u> Let Z^{n-q} be a manifold with Z/p-singularity $Z_1/_{Z/p}$, let X^n be a manifold, and let $f: Z^{n-q} \to X^n$.

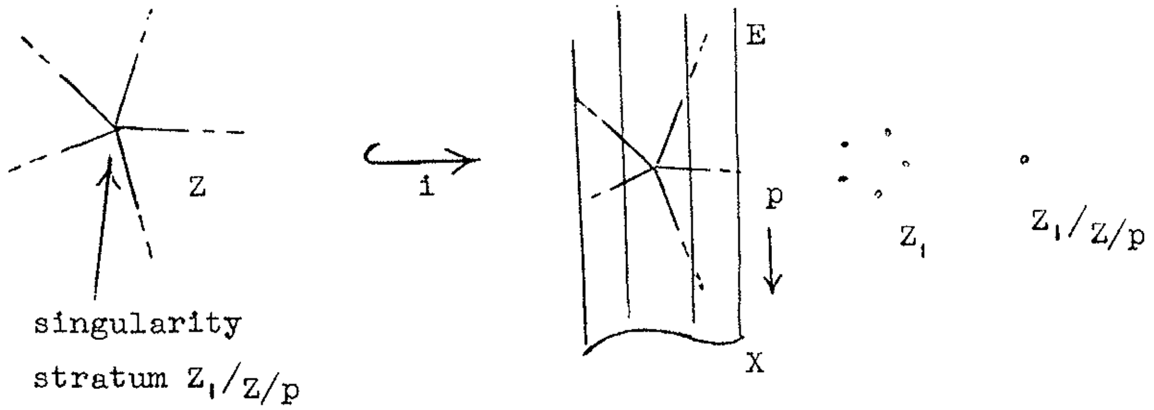
(Note that Z^{n-q} is not required to have a U-structure;e.g. it need not be a ${}^{(1)}V$-manifold if X^n is not a U-manifold.)

A <u>${}^{(1)}V$-orientation</u> for f is an equivalence class of factorisations of f:

$$Z \xrightarrow{i} E \xrightarrow{p} X$$

where $p: E \to X$ is a (stable) complex vector bundle over X and

and i is an embedding carrying a complex structure on its normal bundle ν_i (away from the singularity set $Z_1/_{Z/p}$); i also carries a complex structure on the normal bundle μ_i to the singularity stratum $Z_1/_{Z/p} \hookrightarrow E$. We require the complex structures on ν_i and μ_i to be compatible in the following sense: regard Z_1 as the boundary of Z, so that ν_i induces a complex structure on the normal bundle to Z_1, require this structure to be preserved by the Z/p-action, and require the induced complex structure on the normal bundle μ_i to $Z_1/_{Z/p}$ to be the same as the given one.



Two factorisations of f, $Z \xrightarrow{i} E \xrightarrow{p} X$, and $Z \xrightarrow{i'} E' \xrightarrow{p'} X$, are <u>equivalent</u> if there is a bundle E'' containing E and E' as summands, such that within E'' i is isotopic to i' compatibly with all the normal structure.

<u>Definition 4.3.8</u> Two proper ${}^{(1)}V$-oriented maps $f_0 : Z_0 \longrightarrow X$, $f_1 : Z_1 \longrightarrow X$ are said to be <u>cobordant</u> if there is a proper ${}^{(1)}V$-oriented map $F : W \rightarrow X \times R$ such that $\varepsilon_i : X \hookrightarrow X \times R$ $(\varepsilon_i(x) = (x, i)$ $i = 0,1)$ are transverse to F and the pull-back of F by ε_i gives f_i. $(i = 0,1)$

Disjoint union as sum makes the set of cobordism classes into a group, which we denote ${}^{(1)}V^*(X)$.

For $Y \hookrightarrow X$ a deformation retract of a regular neighbourhood we define the relative group ${}^{(1)}V^*(X, X \setminus Y)$ to be the group obtained by using ${}^{(1)}V$-oriented maps with image $\hookrightarrow Y$ and cobordisms with image $\hookrightarrow Y$.

${}^{(1)}V^*(-)$ is a contravariant functor on the category of manifolds: let $x \in {}^{(1)}V^*(X)$ be represented by $f : Z \to X$ and let $g : Y \to X$ be a map of manifolds.

$$\begin{array}{ccc} & & Z \\ & & \Big\downarrow f \quad {}^{(1)}V\text{-oriented} \\ Y & \xrightarrow{\;g\;} & X \\ \text{manifold} & & \text{manifold} \end{array}$$

We approximate g by a map transverse to f by regarding Z as a manifold with boundary Z_1 and using the standard transversality technique of Thom. The pull-back of Z over Y then gives an element of ${}^{(1)}V^*(Y)$, as required, and the construction is clearly natural.

<u>Theorem 4.3.9</u> ${}^{(1)}V^*(-)$ is a cohomology theory and is dual to ${}^{(1)}V_*(-)$.

<u>Proof</u> It is sufficient to prove the theories are Spanier--Whitehead dual as this implies ${}^{(1)}V^*(-)$ is a cohomology theory on finite complexes. Thus we must show that for X embedded in S^n as the deformation retract of a regular neighbourhood there is a natural isomorphism:-

$$\phi : {}^{(1)}V^*(S^n, S^n \setminus X) \xrightarrow{\cong} {}^{(1)}V_{n-*}(X)$$

The definition of ϕ and ϕ^{-1} proceed exactly as in 4.1.4 except that at each stage we have to deal first with the singularity set $M_1/_{Z/p}$ and then use a regular neighbourhood indexed by I to extend to the rest of M (as in 4.3.6). We omit the details. ▮

$S^{2n-1} * Z/p \big/ Z/p \xrightarrow{i} S^{2n+1}/Z/p$ is a ${}^{(1)}V$-oriented map, for:

(i) Excising the singularity set $(S^{2n-1}/Z/p)$ leaves an open disc $D^{2n} \xrightarrow{i} S^{2n+1}/Z/p$ which has trivial normal bundle ν_i.

(ii) The singularity set $S^{2n-1}/Z/p$ has a normal bundle μ_i in $S^{2n+1}/Z/p$ with a natural complex structure and when this bundle is pulled back to S^{2n-1} it is trivial, (since μ_i is induced by $S^{2n-1}/Z/p \rightarrow S^{2n-1}/S^1 \hookrightarrow CP^\infty$), so that μ_i is compatible with ν_i.

Thus, as anticipated in 4.2, the map i represents an element $\alpha^{(n)} \in {}^{(1)}V^1(S^{2n+1}/Z/p)$. Further, $i^*\alpha^{(n)} = \alpha^{(n-1)}$ so we obtain an element $\alpha_V \in \varprojlim_n {}^{(1)}V^1(S^{2n+1}/Z/p)$. Similarly $S^{2n-1}/Z/p \xrightarrow{i} S^{2n-1}/Z/p$ (which has no singularities) defines an element $\beta_V \in \varprojlim_n {}^{(1)}V^2(S^{2n+1}/Z/p$.

However, ${}^{(1)}V^*(-)$ is not a Z/p-theory (3.1.1) as it has no multiplicative structure. (We shall see in Chapter 5, when we calculate ${}^{(1)}V^*$, that it contains elements of order p^2 and yet $1 \in {}^{(1)}V^0$ has order p). Also, though we shall not prove it here, ${}^{(1)}V^*(X)$ does not map onto $H^*(X;Z/p)$ for all X; to adjust ${}^{(1)}V^*(-)$ to be a multiplicative theory, and thus to have all the machinery of a representative Z/p-theory (3.3.1) we need to introduce 'product elements' with 'join-singularities'

4.4 Products of elements of ${}^{(1)}V^*$

Step by step, we now construct ${}^{(\infty)}V^*(-)$, the universal (commutative) multiplicative theory generated by the elements of ${}^{(1)}V^*(-)$. It will be a representative Z/p-theory and in due course we shall show that a sub-theory $V^*(-) \hookrightarrow {}^{(\infty)}V^*(-)$ gives a natural geometric realisation of the universal Z/p--formal group as $V^*(B\ Z/p)$.

Suppose $A_1, A_2, \ldots, A_n \in {}^{(1)}V^*$.

The Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is a pseudo-manifold with link classes $\underbrace{Z/p * \ldots * Z/p}_{r\text{-fold join}} * S^{m-r}$ $(r \leq n)$. A neighbourhood of each singularity stratum has fibre $Z/\underbrace{p * \ldots * Z}_{r}/p * pt.$ and structure group $(Z/p)^r$;(when all the deeper strata have been excised, in order to make the neighbourhood a fibre bundle.)

In the picture obtained by 'cutting along the singularities' this is seen more clearly: consider A_i as a U-manifold with boundary ∂A_i and a free Z/p-action t_i on ∂A_i, and then consider the Cartesian product $A = A_1 \times \ldots \times A_n$. The boundary of A, ∂A, has faces and corners (in the sense of Baas (3)) viz:

$$\partial A = B_1 \vee B_2 \vee \ldots \vee B_n \quad (\text{where } B_1 = \partial A_1 \times A_2 \times \ldots \times A_n$$

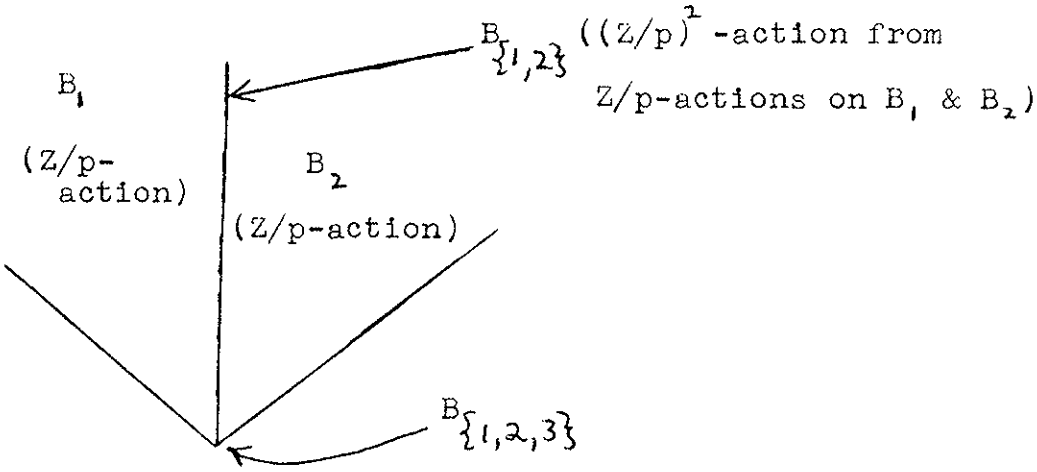
$$B_2 = A_1 \times \partial A_2 \times \ldots \times A_n \text{ etc.})$$

and the B_i are manifolds with corners:

$$\partial B_1 = B_{\{1,2\}} \vee \ldots \vee B_{\{1,n\}} \quad (\text{where } B_{\{1,2\}} = \partial A_1 \times \partial A_2 \times A_3 \times \ldots \times A_n \text{etc.})$$

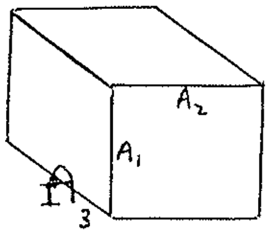
etc. in fact giving manifolds with corners B_S indexed by all subsets S of $\{1, \ldots, n\}$.

Further, we have a free action of Z/p on each B_i (from that on ∂A_i) and a free action of $(Z/p)^2$ on each $B_{\{i,j\}}$(from that on $\partial A_i \times \partial A_j$)etc.



Quotienting this picture by all the group actions takes us back to the picture of a pseudo-manifold with links of points $\underbrace{Z/p * \ldots * Z/p}_{r} * S^{m-r}$.

We would like a bordism relation on our products of elements $A_i \in {}^{(i)}V^*$ which will give $A_1 \times \ldots \times A_n$ bordant to $A_{\sigma(1)} \times \ldots \times A_{\sigma(n)}$ (at least up to sign) for each permutation σ of $\{1, 2, \ldots, n\}$.

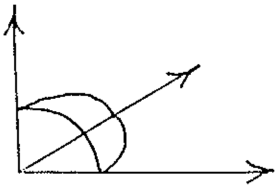


~~If we are to use the manifold $(A_1 \times \ldots \times A_n) \times I$ as a bordism~~ To obtain an isomorphism of $A_1 \times \ldots \times A_n$ to $A_{\sigma(1)} \times \ldots \times A_{\sigma(n)}$ ~~then~~ we must 'forget' the labelling data which orders the faces and corners of $A_1 \times \ldots \times A_n$ (or, equivalently, forget the labels on the Z/p-singularities and their intersections). The next section sets up a framework for this:- 'manifolds with unlabelled corners'.

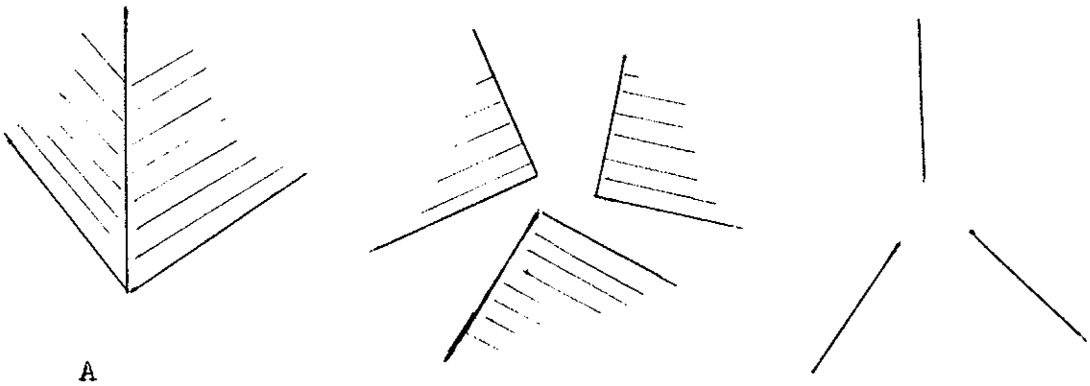
4.5 Douady's Manifolds with Corners

Definition 4.5.1 (Douady (16)) A C^{∞}-manifold with corners is defined just like a C^{∞}-manifold except that as 'model space' we allow sectors of R^n of the form:-

$A = \{x_1 \geqslant 0,\ x_2 \geqslant 0, \ldots,\ x_k \geqslant 0\} \subset R^n$ (where x_i are orthogonal axes)



Definition 4.5.2 A j-face of the sector A (above) is a subset of A with the further restrictions $\{x_{i_1} = 0, \ldots, x_{i_j} = 0\}$ $(i_1, \ldots, i_j \in \{1, 2, \ldots, k\})$ (so A has $\binom{k}{j}$ j-faces)



A

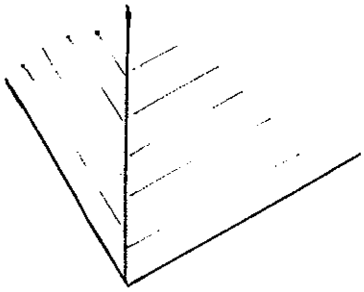
1-faces of A

2-faces of A

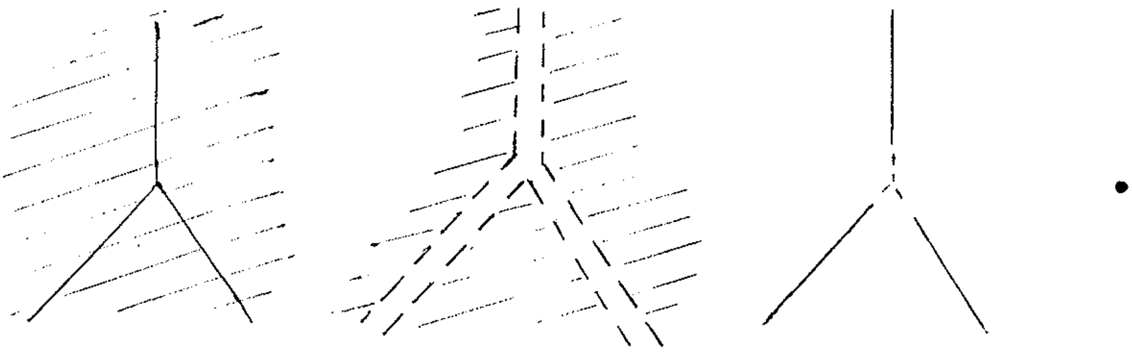
Remark 4.5.3 If M is a C^{∞}-manifold with corners then the points corresponding to points in j-faces of the model sectors give rise to well-defined strata in M for each j; (just as a manifold modelled on 'half-space' has a well-defined boundary) See Douady (16) for the technical details.

Definition 4.5.4 The index of $x \in$ sector A is defined to be the greatest codimension of faces of A containing x. If M is a C^{∞}-manifold with corners define M_k to be those points of index k. (Well-defined by the above remark). M_k is a manifold.

Thus a C^∞-manifold with corners M has boundary ∂M stratified by the manifolds M_i :-



M



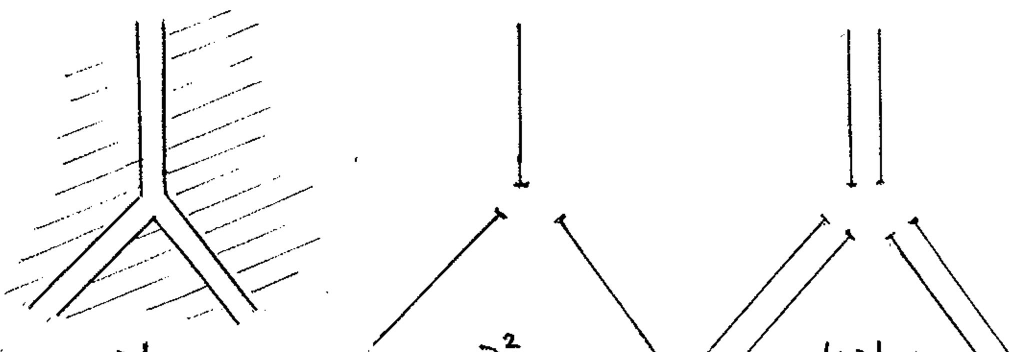
$\partial M \quad = \quad M_1 \quad \cup \quad M_2 \quad \cup \quad M_3$

Each point $x \in M$ has a neighbourhood diffeomorphic to a well-defined sector A of R^n; thus it has a well-defined 'tangent sector' A_x .

Definition 4.5.5 (Douady)

$\partial^k M = \{(x,F);\ x \in M_k \cup M_{k+1} \cup \dots\ \&\ F \text{ a k-face of } A_x \text{ (tangent sector)}\}$

Then $\partial^k M$ is a C^∞-manifold with corners. (For proof see Douady(16))

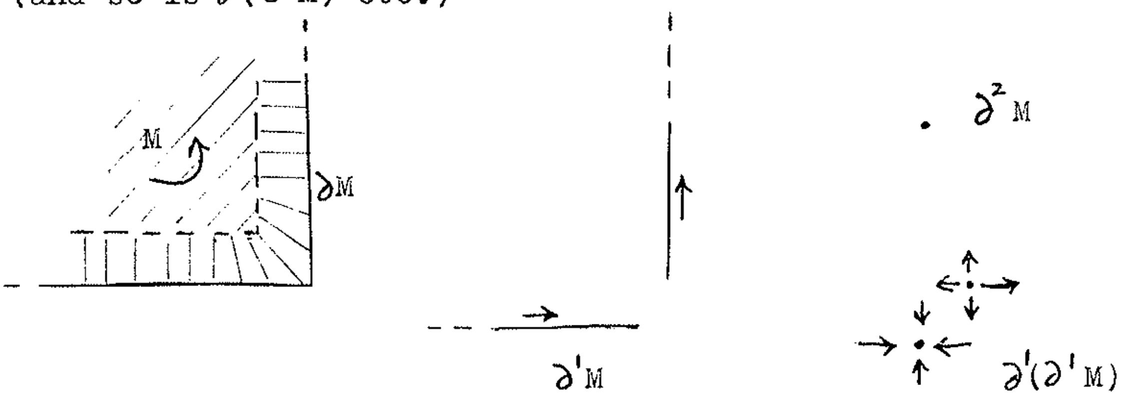


$\partial^1 M \qquad \partial^2 M \qquad \partial^1(\partial^1 M)$

Remark 4.5.6 We observe that in general $\underbrace{\partial^1(\partial^1 \dots (\partial^1}_{r} M)$ is an $r!$ -cover of $\partial^r M$. (and that M_r is the interior of $\partial^r M$.)

<u>Definition 4.5.7</u> A <u>U-manifold with corners</u> is a C^{∞}-manifold with corners M together with a complex structure on the stable tangent bundle to the interior of M.

Being the boundary of M, ∂M has a collar in M and so the complex structure on τM induces one on τM_{1} (c.f. 4.3.1) and thus $\partial' M$ is a U-manifold with corners in a natural way (and so is $\partial'(\partial' M)$ etc.)

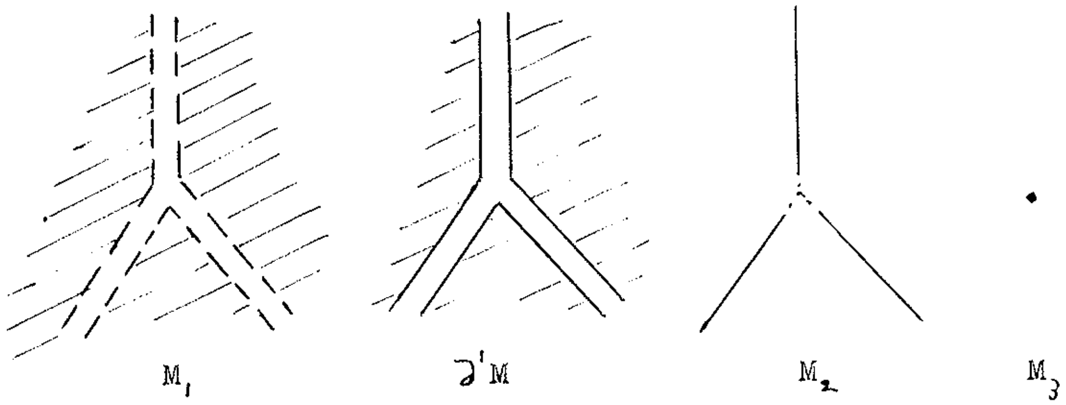


(the arrows represent the U-orientations)

Note that $\partial^2 M$ has no naturally induced U-orientation as the orientations induced by the two 'sides' of $\partial^2 M \hookrightarrow \partial M$ are opposite as the 'inward normals' are opposite. However $\underbrace{\partial'(\partial' \ldots (\partial'}_{r} M))$ always has a natural U-orientation and it is clear that this U-orientation is reversed by the odd permutations in Σ_r and preserved by the even permutations. (Recall $\underbrace{\partial'(\partial' \ldots (\partial'}_{r} M))$ is the r! -cover of $\partial^r M$, and the action of Σ_r corresponds to reordering the labels of the faces of M meeting at $\partial^r M$)

'U-manifolds with corners' will give a suitable generalisation of 'products of elements of ${}^{(1)}V^*$' once we have put appropriate actions of Z/p on M, etc. (In fact the Douady manifold with corners is a manifold with $\{e\}, \{e\} * \{e\}, \ldots$ singularities (so having links S^n, $S^{n-1} * \{e\}$ etc.) where e is the identity group.)

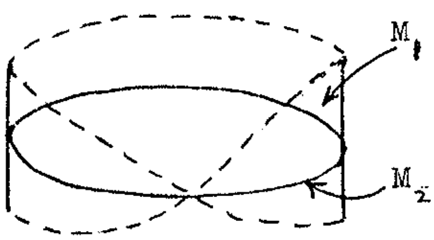
We now examine, informally, the group actions necessary on the faces and corners of a U-manifold with corners to define a pseudo-manifold with links S^n, $S^{n-1}*Z/p$, $S^{n-2}*Z/p*Z/p, \ldots$.



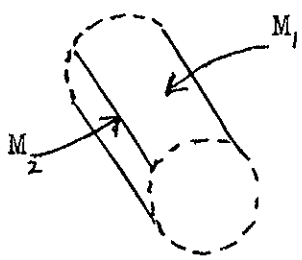
First, we must have a free action of Z/p on M_1 (which corresponds to the stratum of link type $S^{n-1}*Z/p$)
This action induces a Z/p-action on $\partial' M$, and hence two Z/p-actions on M_2 (one from each 'side'). In order to obtain a stratum of link type $S^{n-2}*Z/p*Z/p$ from M_2 we require that these two Z/p-actions be "orthogonal" i.e. that locally they give a free$(Z/p)^2$-action.
Similarly we get three induced Z/p-actions on M_3 and we require that locally they give a free $(Z/p)^3$-action, etc.

The reason we only have a local action of $(Z/p)^2$ on M_2 is that the two "sides" of M_2 in M_1 may not be globally labelled, i.e. they may interchange:-

Examples



(Möbius strip)



These correspond to the situation where a Z/p-singularity set (corresponding to M_1), intersects itself in a $Z/p*Z/p$-singularity set (corresponding to M_2).

The effect of allowing such self-intersections of singularities is to increase the structure group on the "depth n" singularity normal bundle (fibre $(\underbrace{Z/p * Z/p * \ldots * Z/p}_{n}) * \text{pt.}$) from $(Z/p)^n$ to $Z/p \wr \Sigma_n$.

Remark 4.5.8 The wreath product $Z/p \wr \Sigma_n$ is defined by the split short exact sequence:-

$$0 \longrightarrow (Z/p)^n \longrightarrow Z/p \wr \Sigma_n \longrightarrow \Sigma_n \longrightarrow 0$$

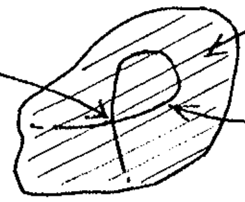
(where $\sigma \in \Sigma_n$ act as the permutations of the factors of $(Z/p)^n$).

A good way to think of $Z/p \wr \Sigma_n$ is as the multiplicative group of n×n permutation matrices (matrices with one entry in each row and one in each column) with entries $e^{2\pi k i/p}$ $(k=1,..,p)$

In fact our free actions of $(Z/p)^n$ defined locally on M_n fit together globally to give a free action of $Z/p \wr \Sigma_n$ on $\underbrace{\partial'(\partial' \ldots (\partial'}_{n} M))$, which is an n! -cover of M_n. (The Σ_n-bundle $\underbrace{\partial'(\partial' \ldots (\partial'}_{n} M)) \twoheadrightarrow M_n$ carries the 'global' data about which singularity intersects which.)

Example of a $Z/2 * Z/2$-singularity with group $Z/2 \wr \Sigma_2$

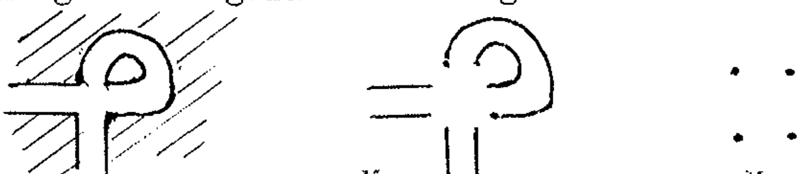
self-intersection of singularity



2-dimensional manifold M

Z/2-singularity

Cutting along the singularities we get:



M_1 M_2

We have a free Z/2-action on M_1 :-

We have a free $(Z/2)^2$-action on M_2:-

(local ≅global in this case)

And we have a free $Z/2 \wr \Sigma_2$-action on $\partial'(\partial' M)$:-

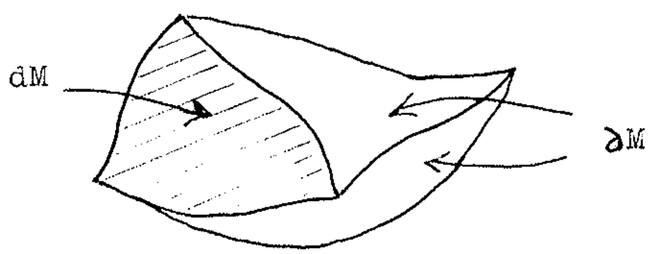
4.6 The Bordism Theories ${}^{(n)}V_*(-)$

We use the ideas of 4.5 to formally define a series of bordism theories, ${}^{(0)}V_*(-) \cong U_*(-) \longrightarrow {}^{(1)}V_*(-) \longrightarrow {}^{(2)}V_*(-) \longrightarrow {}^{(3)}V_*(-) \ldots$.

Definition 4.6.1 An ${}^{(n)}$V-manifold M is a U-manifold with corners, all having index $\leq n$, together with a free Z/p-action t on M, (preserving the U-orientation) which induces free $(Z/p)^r$-actions locally on the corners M_r (and hence free $Z/p \wr \Sigma_r$-actions on $\underbrace{\partial'(\partial' \ldots (\partial'}_{r} M))$.)

Remark 4.6.2 Quotienting by all the group actions gives an equivalent picture of an ${}^{(n)}$V-manifold as a pseudo-manifold with points having links $S^{m-1}, S^{m-2} * Z/p, S^{m-3} * Z/p * Z/p, \ldots, S^{m-n-1} * \underbrace{Z/p * \ldots * Z/p}_{n}$ where the stratum with link type $S^{m-r-1} * \underbrace{Z/p * \ldots * Z/p}_{r}$ has normal bundle with structure group $Z/p \wr \Sigma_r$ (acting on fibre $\underbrace{Z/p * \ldots * Z/p}_{r} * pt$) and where the stratum with links S^{m-1} has a complex strucure on its stable tangent bundle preserved by the group actions on its boundary etc. From this picture we return to that of 4.6.1 by 'cutting along all the singularities.'

The relative version of a U-manifold with corners is a U-manifold with corners and boundary (M,dM) which we define to be a manifold M with corners, having boundary $dM \cup \partial M$, such that dM is a manifold with corners (its faces and corners being $\partial M \cap dM$).



Thus an ${}^{(n)}$V-manifold with boundary is defined to be a U-manifold with corners and boundary (M,dM) together with appropriate group actions on the strata of ∂M (and then dM is an ${}^{(n)}$V-manifold).

(2) $\underbrace{\partial'(\partial'\ldots(\partial'}_{n}M))$ has a collar in $\underbrace{\partial'(\partial'\ldots(\partial'}_{n-1}M))$. We use this to extend f_1 to a map f_2 defined on the two deepest strata $\left(\underbrace{\partial'(\partial'\ldots(\partial'}_{n}M))\big/ Z/p\wr\Sigma_n\right)\cup\left(\underbrace{\partial'(\partial'\ldots(\partial'}_{n-1}M))\big/ Z/p\wr\Sigma_{n-1}\right)$ and homotopic to f.

(3) Keeping f_2 fixed on the deepest stratum we homotop f_2 to f_3 transverse to $\{\frac{1}{2}\}$ on $\partial'(\partial'\ldots(\partial'M))\big/ Z/p\wr\Sigma_{n-1}$ (using Sard)

.... repeating process, at each stage using collar of $\underbrace{\partial'(\partial'\ldots(\partial'}_{r}M))$ in $\underbrace{\partial'(\partial'\ldots(\partial'}_{r-1}M))$, until we arrive at an f' defined on the whole of M and homotopic to f, with f' transverse at $\{\frac{1}{2}\}\hookrightarrow I$.

As in 4.3.6 ,once we have this transversality theorem the geometric methods of Bröcker & tom Dieck (6) generalise immediately to prove the Eilenberg-Steenrod axioms for ${}^{(n)}V_*(-)$. We omit the details. ▮

As in 4.3.7 we may use Quillen's duality approach to define ${}^{(n)}V$-oriented maps and thus define a cobordism theory ${}^{(n)}V^*(-)$. To prove that ${}^{(n)}V^*(-)$ is indeed the dual cohomology theory to ${}^{(n)}V_*(-)$ is just a matter of generalising 4.3.9 (using induction and the collars of 4.6.5)

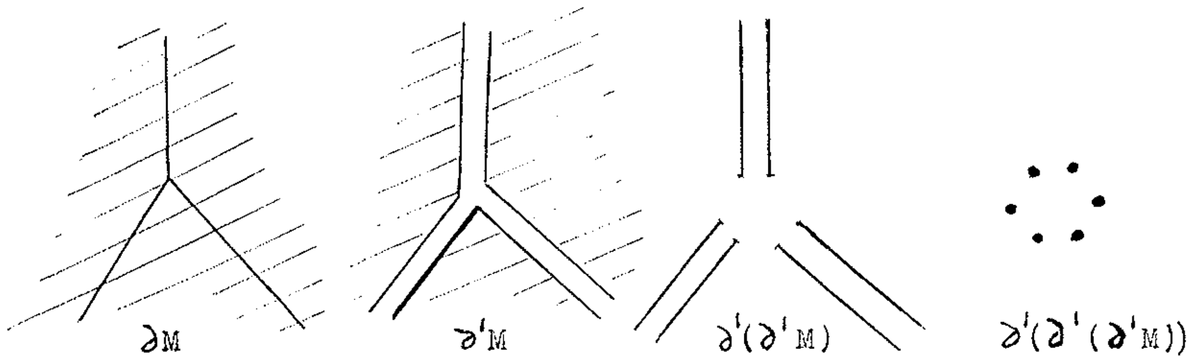
<u>Remark 4.6.6</u> Cartesian product of ${}^{(n)}V$-manifolds induces a series of multiplications ${}^{(m)}V^*(-)\times{}^{(n)}V^*(-)\rightarrow{}^{(m+n)}V^*(-)$. Since the 'orientation' of an ${}^{(n)}V$-manifold is induced by the U-structure on its top dimensional stratum these multiplications are commutative, (in the sense of Milnor i.e. $a.b=(-1)^{\dim a.\dim b}b.a$) (see 4.4)

<u>Definition 4.6.3</u> Two $^{(n)}V$-manifolds M', M'' are said to be <u>bordant</u> if there is an $^{(n)}V$-manifold with boundary (M, dM) such that $dM \cong M' \amalg (-M'')$. Under disjoint union as sum bordism classes of compact $^{(n)}V$-manifolds form a group, denoted $^{(n)}V^*$. The bordism theory $^{(n)}V_*(-)$ is defined in the analagous way to 4.3.5.

<u>Remark 4.6.4</u> $^{(0)}V_*(-)$ is just $U_*(-)$. Also the definition of $^{(n)}V_*(-)$ agrees with our earlier definition of $^{(1)}V_*(-)$ for $n=1$.

<u>Theorem 4.6.5</u> $^{(n)}V_*(-)$ is a generalised homology theory.

<u>Proof</u> The geometric proof of Bröcker & tom Dieck that $N_*(-)$ is a homology theory generalises to $^{(n)}V_*(-)$ as it did for $^{(1)}V_*(-)$ in 4.3.6. Again the only difficulty is to prove a transversality theorem for maps $f: M \to I$ (M an $^{(n)}V$-manifold); to show that such an f has an f' transverse to $\{\frac{1}{2}\} \hookrightarrow I$ homotopic to f and close to f we proceed inductively on the strata of M:-



∂M | $\partial' M$ free Z/p-action | $\partial'(\partial' M)$ free $Z/p \wr \Sigma_2$-action | $\partial'(\partial'(\partial' M))$ free $Z/p \wr \Sigma_3$-action

(Regard M as a manifold with singularities and recall that 'transverse' means 'simultaneously transverse on all strata')

The method of constructing f' is:

(1) On the deepest singularity stratum $\underbrace{\partial'(\partial' \ldots (\partial'}_{n} M)) / Z/p \wr \Sigma_n$ we homotop f to f_1, transverse to $\{\frac{1}{2}\}$ (using Sard's theorem)

<u>Definition 4.6.7</u> ${}^{(\infty)}V_*(-)$ is the theory obtained from 4.6.1 by omitting the restriction $r \leqslant n$.

It is obvious geometrically that an element of ${}^{(\infty)}V_m(X)$ cannot have corners of index $> m$ so it is immediate that ${}^{(\infty)}V_*(X) = \varinjlim_n {}^{(n)}V_*(X)$ (and so, being a limit of homology theories, ${}^{(\infty)}V_*(-)$ is also a homology theory).

${}^{(\infty)}V^*(-)$ is a multiplicative theory and the multiplication is commutative (from 4.6.6). Also $V^0(\text{pt.}) = Z/p$, so ${}^{(\infty)}V^*(-)$ is a Z/p-theory (3.1.1). In fact we have:-

<u>Proposition 4.6.8</u> The classes $\begin{pmatrix}\alpha_V\\ \beta_V\end{pmatrix} \in \begin{pmatrix}\varprojlim {}^{(1)}V^1(S^{2n+1}/Z/p)\\ \varprojlim {}^{(1)}V^2(S^{2n+1}/Z/p)\end{pmatrix}$ of 4.2 map to classes $\begin{pmatrix}\alpha_V\\ \beta_V\end{pmatrix} \in \begin{pmatrix}\varprojlim {}^{(\infty)}V^1(S^{2n+1}/Z/p)\\ \varprojlim {}^{(\infty)}V^2(S^{2n+1}/Z/p)\end{pmatrix}$ which make ${}^{(\infty)}V^*(-)$ into a representative Z/p-theory (3.3.1)

<u>Proof</u> This follows at once from the geometric definition of α_V, β_V in 4.2. There is just the small technical point that 3.3.1 asked for $\alpha_V, \beta_V \in {}^{(\infty)}V^*(BZ/p)$. The best way to deal with this is to observe that Rourke's theorem and its corollary (3.1.2 and 3.1.3) only require $\alpha_V, \beta_V \in \varprojlim_n {}^{(\infty)}V^*(S^{2n+1}/Z/p)$ (as his proof of 3.1.2 only uses finite skeleta of BZ/p). As 3.1.2 is satisfied ${}^{(\infty)}V^*(-)$ has a spectrum weakly equivalent to a product of Eilenberg-Maclane spectra $K(\underset{\sim}{Z/p})$ (Rourke (34)). We may now use this product of $K(\underset{\sim}{Z/p})$'s to define ${}^{(\infty)}V^*(-)$ on infinite CW complexes, and we have ${}^{(\infty)}V^*(BZ/p) \cong \varprojlim_n {}^{(\infty)}V^*(S^{2n+1}/Z/p)$, resolving the technicality. ▌

<u>Remark 4.6.9</u> It would be interesting to determine the structure of the bordism ring of U-manifolds with corners (i.e. ${}^{(\infty)}V^*$ with Z/p replaced by the trivial group $\{e\}$.(Recall 4.5.7))

4.7 The Exact Triangle

Theorem 4.7.1 For all $n > 0$ the following triangle of homology theories is exact for all CW complexes X.

$$\begin{tikzcd}
{}^{(n-1)}V_*(X) \arrow[rr, "i_n"] & & {}^{(n)}V_*(X) \arrow[dl, "j_n"] \\
& \pi_*(X^+ \wedge (\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+)) \arrow[ul, "k_n"] &
\end{tikzcd}$$

(i_n degree 0
j_n degree $-n$
k_n degree $n-1$)

('+' denotes 'with a disjoint base point')

Proof First an explanation of $\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)$:-
This denotes the spectrum obtained from $\underset{\sim}{MU} \wedge E(Z/p \wr \Sigma_n)^+$ by quotienting by the action of $Z/p \wr \Sigma_n$, where the action on $\underset{\sim}{MU}$ is given by $Z/p \wr \Sigma_n \longrightarrow \Sigma_n \xrightarrow{sign} \Sigma_2$ and Σ_2 acting on $\underset{\sim}{MU}$ by reversing the U-orientation.

Thus $\pi_*(X^+ \wedge (\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+))$ is the bordism theory of U-manifolds which are the total space of principal $Z/p \wr \Sigma_n$--bundles with the further property that $Z/p \wr \Sigma_n$ acts on the U-orientation via $Z/p \wr \Sigma_n \twoheadrightarrow \Sigma_n \longrightarrow \Sigma_2$. (Of course such bundles are in one to one correspondance with associated bundles having fibre $Z/p\underbrace{* \ldots *Z/p}_{n \text{ fold join}}$.)

The heart of the proof that the triangle is exact is to regard ${}^{(n)}V_*(-)$ as bordism of ${}^{(n-1)}V$-manifolds with boundary, where the boundary is the total space of a bundle with fibre $Z/p\underbrace{* \ldots *Z/p}_{n \text{ fold join}}$ and structure group $Z/p \wr \Sigma_n$. This is done by taking the 'manifolds with singularities' picture of ${}^{(n)}V_*(-)$ and "cutting along" just the "depth n" singularity set (which is a codimension n submanifold) to leave a boundary which is the total space of a $Z/p* \ldots *Z/p$-bundle (with the appropriate U-orientation structure).

(i) <u>Definition of maps in the triangle</u>

i_n is clear.

j_n picks out the 'depth n' singularity stratum and its normal structure bundle $(\underbrace{\partial'(\partial' \ldots (\partial'}_{n} M))$ in the notation of 4.5)

k_n takes a $Z/p \wr \Sigma_n$-bundle which is a U-manifold to the total space of the associated bundle with fibre $\underbrace{Z/p * \ldots * Z/p}_{n \text{ fold join}}$. (which is an ${}^{(n-1)}V$-manifold).

(ii) <u>Exactness of the triangle</u>

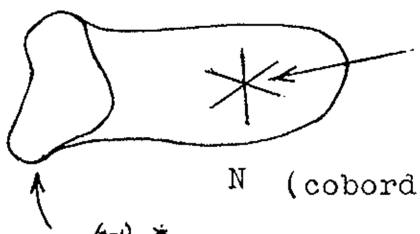
$$\begin{tikzcd} 1 \arrow[rr, "i"] & & 2 \arrow[dl, "j"] \\ & 3 \arrow[ul, "k"] & \end{tikzcd}$$

(The proofs will be given for

$$\begin{tikzcd} {}^{(n-1)}V^* \arrow[rr] & & {}^{(n)}V^* \arrow[dl] \\ & \pi_*(\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+) \arrow[ul] & \end{tikzcd}$$

but will extend immediately to the bordism of any X)

1 : By definition ker i = elements of ${}^{(n-1)}V^*$ that cobord to zero if we allow depth n singularities. Let $M \in$ ker i :-



depth n singularity

N (cobordism of M to zero in ${}^{(n)}V^*$)

$M \in {}^{(n-1)}V^*$

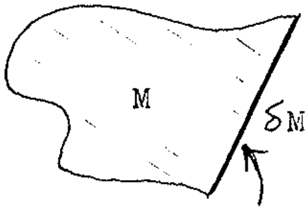
Cutting along the depth n singularity in N gives us a cobordism in ${}^{(n-1)}V^*$ of M to the total space of a $Z/p * \ldots * Z/p$--bundle. i.e. ker i $\hookrightarrow$ Im k.

Conversely, given $M \in$ Im k, M is the total space of a $Z/p * \ldots * Z/p$-bundle, and then M*base is a cobordism of M to zero in ${}^{(n)}V^*$. i.e. Im k $\hookrightarrow$ ker i.

(2) : ker j = elements of $^{(n)}V^*$ that have a cobordism to zero of the depth n singularity stratum and its normal bundle.

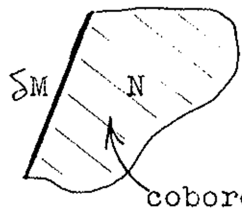
Use the picture explained above of $^{(n)}V^*$ as elements of $^{(n-1)}V^*$ with boundary, where the boundary is the total space of a $\mathbb{Z}/p\underbrace{*\ldots*}_{n}\mathbb{Z}/p$-bundle (and regard cobordisms in $^{(n)}V^*$ similarly).

Let $M \in \ker j$



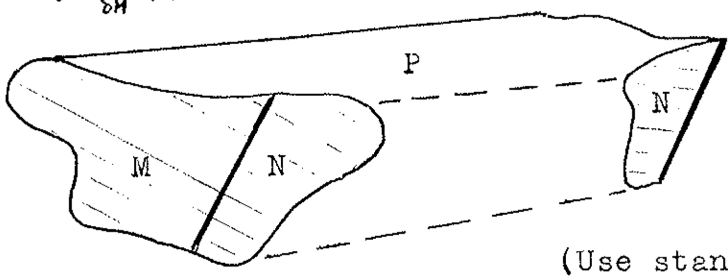
boundary = total space of $\mathbb{Z}/p\underbrace{*\ldots*}_{n}\mathbb{Z}/p$-bundle

δM



cobordism of δM to zero as a $\mathbb{Z}/p\underbrace{*\ldots*}_{n}\mathbb{Z}/p$-bundle

Consider the manifold P formed from $(M \cup_{\delta M} N) \times I$:-



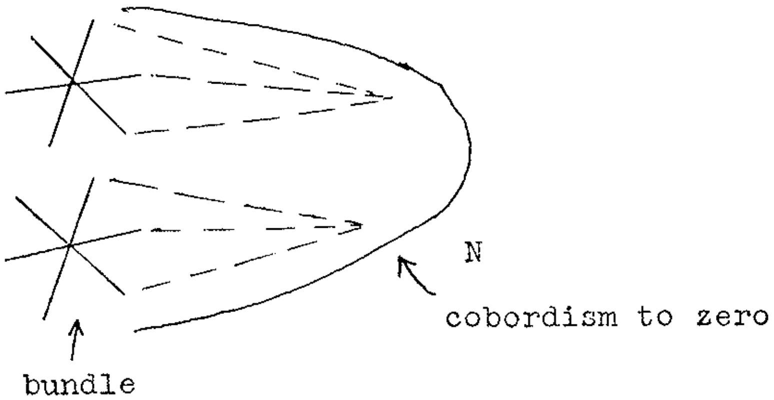
"pushed around corner to side of P"

(Use standard angle-straightening techniques of e.g. Conner & Floyd ([illegible]) for this

The copy of N on the "side" of P is the total space of a $\mathbb{Z}/p\underbrace{*\ldots*}_{n}\mathbb{Z}/p$-bundle and so may be regarded as a depth n singularity in P; thus P gives a cobordism in $^{(n)}V^*$ of M to $M \cup_{\delta M} N$ i.e. a cobordism of M to an element in Im i .

i.e. $\ker j \hookrightarrow \operatorname{Im} i$ (Im i $\hookrightarrow$ ker j is trivial to prove)

(3): $\ker k = \underbrace{Z/p*\dots*Z/p}_{n}$- bundles whose total space cobords to zero as an element of ${}^{(n-1)}V^*$.



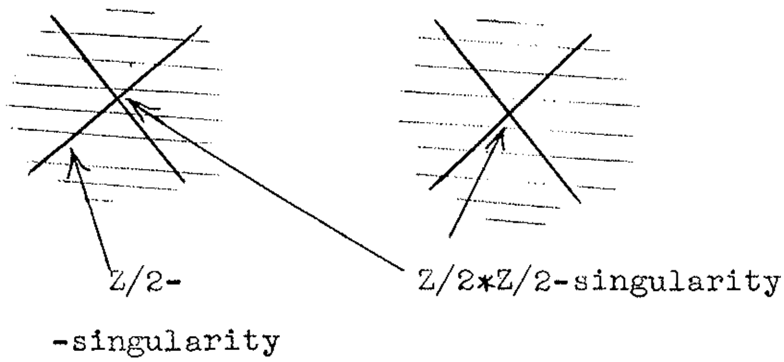
Using the description of elements of ${}^{(n)}V^*$ we used for (2) N represents an element of ${}^{(n)}V^*$ which has the given $\underbrace{Z/p*\dots*Z/p}_{n}$-bundle as a neighbourhood of its depth n singularity set. i.e. $\ker k \hookrightarrow \operatorname{Im} j$.

Exactly the same diagram has an obvious interpretation to prove $\operatorname{Im} j \hookrightarrow \ker k$, completing the proof of the theorem. ▮

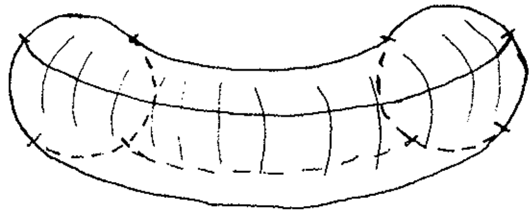
Remark 4.7.2 One should be able to adapt this proof to prove that we have a corresponding cofibration triangle of spectra, which would give an alternative proof that the theories ${}^{(n)}V_*(-)$ are homology theories (by induction).

Remark 4.7.3 In the proof of 4.7.1 exactness at (2) is the most difficult to visualise, so we give an example:-

Let M be



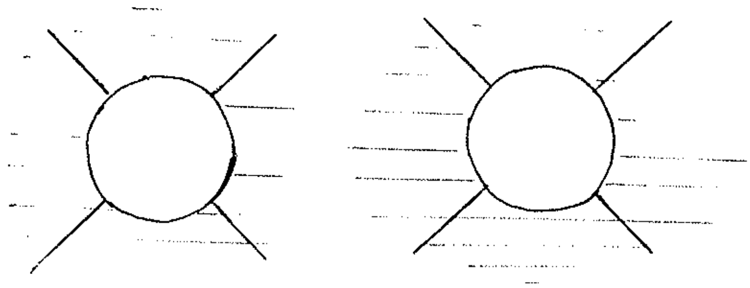
The $Z/2*Z/2$-singularity in M cobords to zero:



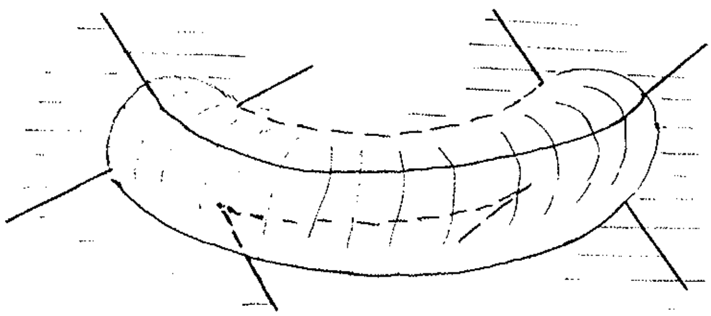
N

We show how N can be used to cobord M to a manifold with just $Z/2$-singularities; the picture of M we need to use is with the $Z/2*Z/2$-singularity 'cut out':

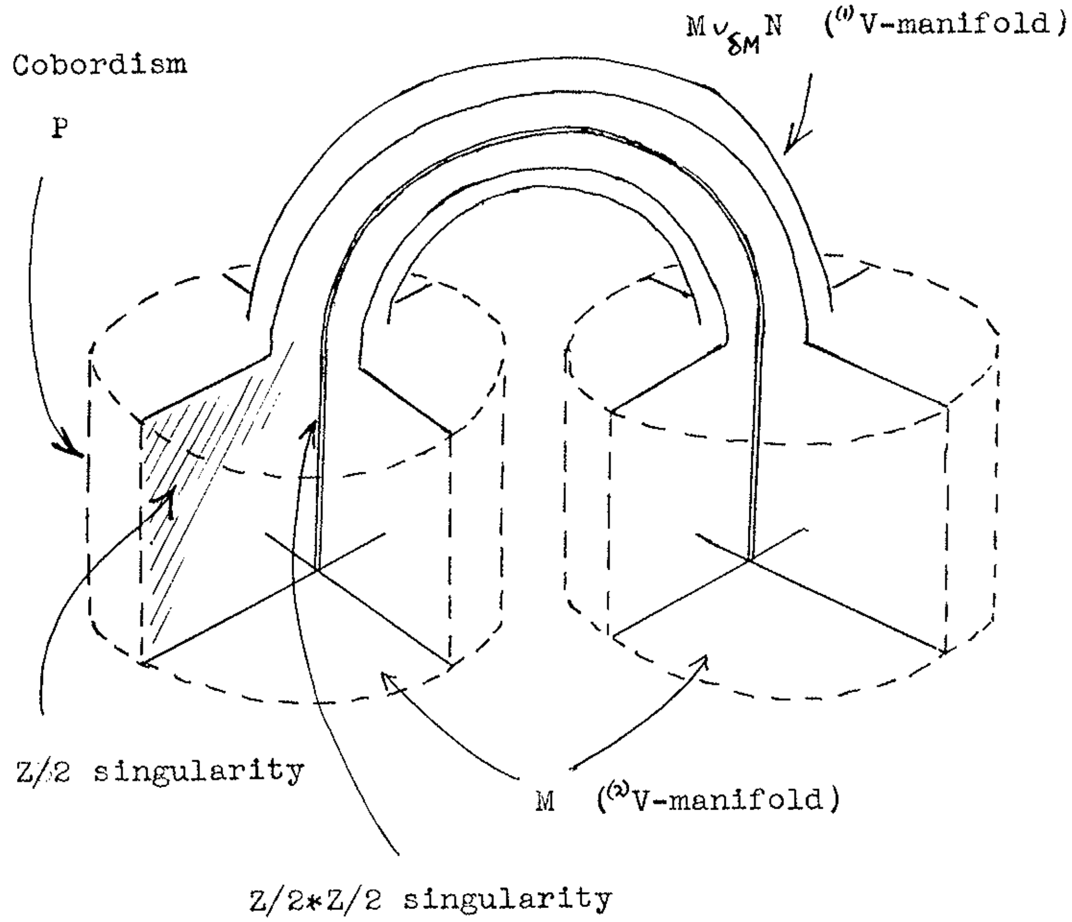
i.e. M is:



Then $M\vee_{\delta M}N$ is the manifold with $Z/2$-singularities:



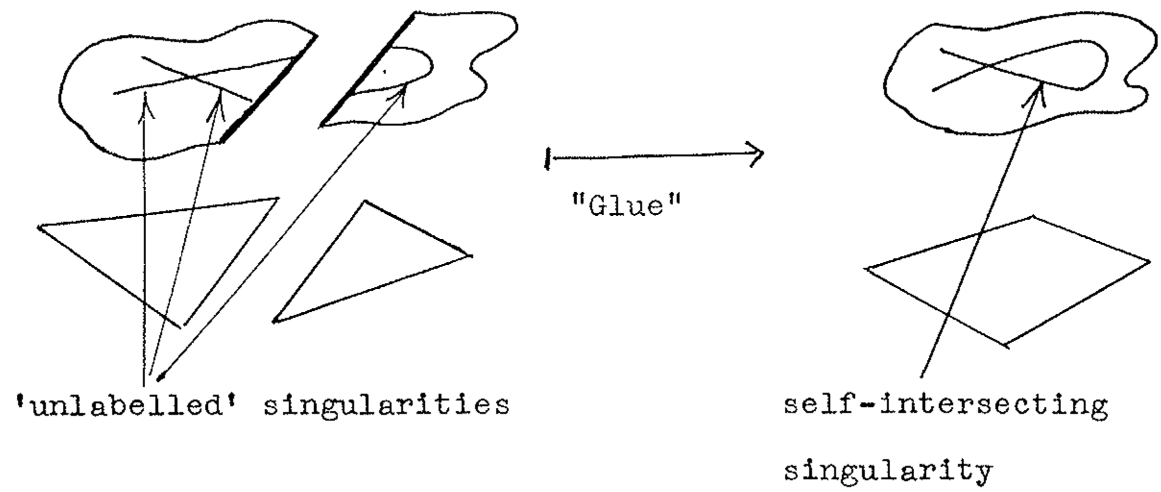
$P = (M\vee_{\delta M}N)\times I$ is a manifold with $Z/2$-singularities, having boundary two copies of $M\vee_{\delta M}N$. For one of these copies we regard N (which is the total space of a $Z/2*Z/2$-bundle) as being a $Z/2*Z/2$-singularity in P. For the other copy we regard N as part of $M\vee_{\delta M}N$, a manifold with $Z/2$-singularities. Thus P represents a cobordism in ${}^{(2)}V^*$ of M to $M\vee_{\delta M}N$ $(\in {}^{(1)}V^*)$. On the next page we picture P as a manifold with $Z/2$ and $Z/2*Z/2$-singularities:-



$M \cup_{\delta M} N$ ($^{(1)}$V-manifold)
Cobordism
P
Z/2 singularity
M ($^{(2)}$V-manifold)
Z/2*Z/2 singularity

4.8 The Mock-Bundle Approach to Cobordism Theories

Having defined ${}^{(n)}V$-manifolds one may define the cohomology theories ${}^{(n)}V^*(-)$ either, as I have done, in Quillen's "proper oriented map" manner, or as cobordism of mock-bundles over X with blocks ${}^{(n)}V$-manifolds (see Rourke & Sanderson (35) for definitions and properties of mock-bundles); mock-bundles have the advantage that there are quick proofs that the geometrically defined theories ${}^{(n)}V^*(-)$ are cohomology theories (one only has to check the 'Suspension','Extension', and 'Glue' axioms of Rourke & Sanderson (35)) and also one has a canonical Δ-spectrum for ${}^{(n)}V^*(-)$. The disadvantage as far as we are concerned is that mock-bundles are defined on P.L. or cell complexes and as we have only to deal with X a manifold it seems neater to avoid complications of cell-decomposition etc. I have compromised by using Quillen's type of definition but only outlining the proofs that the theories are cohomology theories (these proofs being clear in principle but tedious in detail); however some extra geometric insight is to be gained from the mock-bundle approach: e.g. consider the "Glue" axiom for cohomology theories (Rourke & Sanderson (35)) which shows us that once 'unlabelled' singularities are allowed in the blocks, self-intersecting singularities must be allowed if the theory is to satisfy "Glue":



5 Calculation of ${}^{(1)}V^*$

In this chapter we calculate ${}^{(1)}V^*$ as an abelian group and as a U^*-module. This is included because it is easily accessible using Conner & Floyd's techniques but in fact we shall not need the results for the following chapters.

Recall that 4.7 with n=1 gives the exact triangle:

$$\begin{tikzcd} U_* \arrow[rr, "i"] & & {}^{(1)}V_* \arrow[dl, "j"] \\ & U_*(BZ/p) \arrow[ul, "k"] & \end{tikzcd}$$

i degree 0

j degree -1

k degree 0

The structure of $U_*(BZ/p)$ as an abelian group and as a U_*-module was determined by Conner & Floyd ((10) & (11)). (Some of their results are only stated for $\Omega_*(BZ/p)$ but their methods give precisely analogous results for $U_*(BZ/p)$. Kamata (22) gives a proof (22) using the formal group law on $U^*(-)$ which is neat algebraically (but less geometric).

5.1 ${}^{(1)}V^*$ as an abelian group

Let x_{2i} be a set of polynomial generators of U^*, and let $\Gamma_*(p)$ denote the subring generated by those x_{2i} with $i \neq p-1$ (so that $U^* \cong \Gamma_*(p)[CP^{p-1}]$).

A set of U^*-module generators for $U_*(BZ/p)$ is given by the classes $y_{2k-1} \in U_{2k-1}(BZ/p)$, where y_{2k-1} denotes the class:-

$$S^{2k-1}/Z/p \hookrightarrow BZ/p$$

<u>N.B.</u> As we wish to follow Conner & Floyd as closely as possible it is necessary to use a different action of Z/p on S^{2k-1} than that employed so far. Thus, for the purposes of defining the y_{2k-1} only, we use the action of Z/p on C^k given by:-

$$(z_1, z_2, z_3, \ldots, z_{p-1}, z_p, z_{p+1}, \ldots) \longmapsto (e^{\frac{2\pi i}{p}} z_1, e^{\frac{4\pi i}{p}} z_2, e^{\frac{6\pi i}{p}} z_3, \ldots, e^{\frac{2(p-1)\pi i}{p}} z_{p-1}, e^{\frac{2\pi i}{p}} z_p, e^{\frac{4\pi i}{p}} z_{p+1}, \ldots$$

(Caution: Kamata uses a standard action on C summed k times instead of this, as we do elsewhere.)

Theorem 5.1.1 (Conner & Floyd (10)§36.5, Kamata (22))

There is an isomorphism

$$\Theta : \sum_{k=0}^{n} \Gamma_{2(n-k)}(p) \Big/ p^{[k/p-1]+1}\Gamma_{2(n-k)}(p) \longrightarrow U_{2n+1}(BZ/p)$$

Θ is given by $\Theta(\gamma_{2(n-k)}) = \gamma_{2(n-k)} \cdot y_{2k+1}$

Proof Conner & Floyd (10) ∎

As $U_{odd} = 0$ and $H_{even}(BZ/p;Z) = 0$ the Atiyah-Hirzebruch spectral sequence $H_*(BZ/p;U^*) \Rightarrow U_*(BZ/p)$ at once gives us that $U_{even}(BZ/p) = U_{even} = U_*$ and our exact triangle becomes the long exact sequence:

$$\cdots \to U_{odd} \xrightarrow{i_1} {}^{(1)}V_{odd} \xrightarrow{j_1} U_{even}(BZ/p) \xrightarrow{k_1} U_{even} \xrightarrow{i_2} {}^{(1)}V_{even} \xrightarrow{j_2} U_{odd}(BZ/p) \xrightarrow{k_2} U_{odd} \to \cdots$$

k_1 takes a Z/p-bundle to its total space so is just '$\times p$' $: U_{even} \to U_{eve.}$

$U_{odd} = 0$ and k_1 injective $\Rightarrow {}^{(1)}V_{odd} = 0$

Thus our long exact sequence becomes the short exact sequence:

$$0 \to U_*/pU_* \xrightarrow{i} {}^{(1)}V_* \xrightarrow{j} U_{*-1}(BZ/p) \to 0 \qquad \ldots\ldots\ldots (1)$$

By 5.1.1 $U_{2n+1}(BZ/p)$ is generated as an abelian group by elements $\gamma_{2(n-m)} \cdot y_{2m+1}$ of order $p^a = p^{[m/p-1]+1}$. Denote by $\langle \gamma_{2(n-m)} \cdot y_{2m+1} \rangle$ the cyclic subgroup generated by this element. Let $D^{2m+2}/\!\sim$ denote the class in ${}^{(1)}V_{2m+2}$ of D^{2m} with its boundary identified under the free Z/p-action. (So that $j(D^{2m+2}/\!\sim) = y_{2m+1}$ in (1).)

We shall know ${}^{(1)}V^*$ as an abelian group, once we determine the order of $\gamma_{2(n-m)} \cdot D^{2m+2}/\!\sim$ and once we can find $\langle \gamma_{2(n-m)} D^{2m+2}/\!\sim \rangle \cap (U_{2n+2}/pU_{2n+2})$ for these give us the group extension (1) .

Recall, the order of $y_{2m+1} \in U_{2m+1}(BZ/p)$ is p^a $(a = [m/p-1]+1)$ (by 5.1.1)

Lemma 5.1.2 The order of $\gamma_{2(n-m)} D^{2m+2}/\!\sim$ is $\begin{cases} p^{a+1} & \text{if } 2m+2 = 2j(p-1) \text{ for some } j \\ p^a & \text{otherwise} \end{cases}$

Proof We shall just give the proof for $D^{2m+2}/\sim$ as the case for $\gamma_{2(n-m)}D^{2m+2}/\sim$ follows at once by the same arguement.

$j(p^a D^{2m+2}/\sim) = p^a y_{2m+1} = 0$
so $p^a D^{2m+2}/\sim \in$ image of i in (1)
and thus $p^{a+1} D^{2m+2}/\sim = 0$

so it suffices to show that $p^a D^{2m+2}/\sim \begin{cases} \neq 0 \in {}^{(1)}V_{2m+2} & \text{if } 2m+2=(2p-2)j \\ = 0 \in {}^{(1)}V_{2m+2} & \text{if not} \end{cases}$

(i) Case of $D^{2(p-1)}/\sim$

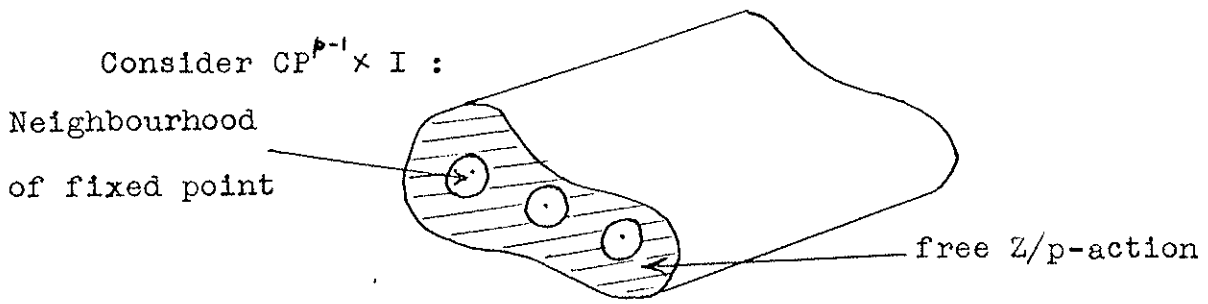
Consider the Z/p-action on CP^{p-1} given by:

$$[z_1, z_2, \ldots z_p] \mapsto [z_1, \rho z_2, \ldots, \rho^{p-1} z_p] \qquad (\rho = e^{2\pi i/p})$$

This action has p fixed points: $[0,\ldots,0,1,0,\ldots,0]$
The normal bundle to each fixed point has a sphere S^{2p-3} with the standard Z/p-action described at the start of 5.1.

(This action is used by Conner & Floyd ((10)§35) to obtain a cobordism of py_{2p-3} to zero in $U_{2p-3}(BZ/p)$. They remove a neighbour--hood of the fixed point set and regard the rest as a cobordism to zero of a free Z/p-manifold. In fact they obtain all the relations in $U_*(BZ/p)$ by this sort of technique)

Consider $CP^{p-1} \times I$:



${}^{(1)}V_*$ is bordism of manifolds with free Z/p- action on the boundary. Thus by regarding the shaded part in the diagram as "free Z/p-action on the boundary of the cobordism" we may interpret the picture as a cobordism in ${}^{(1)}V_*$ of $pD^{2p-2}/\sim$ to CP^{p-1} .
(We know $CP^{p-1} \neq 0 \in U_*/pU_* \hookrightarrow {}^{(1)}V_*$)

(ii) Case of $D^{2n}/\sim$ with $2n < 2p-2$

Conner & Floyd show in their proof ((10)§36.1) that there is a manifold V^{2n} with Z/p-action having p fixed points and standard Z/p-actions on the spheres around these fixed points. (V^{2n} is constructed by interpreting a cobordism to zero of py_{2n-1} in $U_{2n-1}(BZ/p)$.) They also show that this V^{2n} lies in pU_{2n} (since otherwise their construction ((10) §35.2) would give an element of $U_{2n+1}(BZ/p)$ of order $> p$.)

Using $V^{2n} \times I$ instead of $CP^{p-1} \times I$ in the picture for (i) we have a cobordism in ${}^{(1)}V^*$ of $pD^{2n}/\sim$ to V_{2n} ($= 0 \in U_*/pU_* \hookrightarrow {}^{(1)}V_*$)

(iii) Case of $D^{2n}/\sim$ for general n

Write $2n = 2j(p-1) + 2m \quad (m < p-1)$

Take V^{2m} as in (ii) and consider $(CP^{p-1})^j \times V^{2m}$ with the diagonal action of Z/p (from those of (i) & (ii)). This has $p^j \times p = p^{j+1}$ fixed points. Now the picture $((CP^{p-1})^j \times V^{2m}) \times I$ gives a cobordism in ${}^{(1)}V^*$ of $p^{j+1}D^{2n}/\sim$ to $(CP^{p-1})^j \times V^{2m}$.

If $m \neq 0$ then by (ii) $V^{2m} = 0 \in (U_*/pU_* \hookrightarrow {}^{(1)}V_*)$
so $p^{j+1}D^{2n}/\sim \; = 0 \in {}^{(1)}V_*$.
If $m = 0$ then by (i) $p^{j+1}D^{2n}/\sim \; = (CP^{p-1})^j \neq 0 \in (U_*/pU_* \hookrightarrow {}^{(1)}V_*)$. ▮

Lemma 5.1.2 allows us to divide the abelian group generator of ${}^{(1)}V^{2n+2}$ into three classes:

(a) The generators of $U_{2n+1}(BZ/p)$ which split in (1) to give elements of ${}^{(1)}V^{2n+2}$. These give $\langle \gamma_{2(n-k)} D^{2k+2}/\sim \rangle$ for $2k+2 \neq 2j(p-1)$

(b) The generators of $U_{2n+1}(BZ/p)$ which when lifted to ${}^{(1)}V^{2n+2}$ in (1) have order increased by p. These give $\langle \gamma_{2(n-k)} D^{2k+2}/\sim \rangle$ for $2k+2 = 2j(p-1)$.

(c) The remaining generators of U_{2n+2}/pU_{2n+2} not included in (b). These are $\Gamma_{2n+2}(p)$ (since the proof of 5.1.2 gives $(CP^{p-1})^j = p^{j+1}D^{2j(p-1)}/\sim$ in ${}^{(1)}V^*$).

Thus by 5.1.1 and 5.1.2 we have:

<u>Theorem 5.1.3</u> There is an isomorphism of abelian groups:-

$$\Theta: \sum_{\substack{k=0 \\ k \neq j(p-1)-1}}^{n} \left\{ \Gamma_{2(n-k)}(p) \Big/ p^{[k/p-1]+1}\Gamma_{2(n-k)}(p) \right\} \oplus \sum_{\substack{k'=j(p-1)-1 \\ 0<k'<n}} \left\{ \Gamma_{2(n-k')}(p) \Big/ p^{[k'/p-1]+2}\Gamma_{2(n-k')}(p) \right\} \oplus \Gamma_{2n+2}(p) \Big/ p\,\Gamma_{2n+2}(p) \cong {}^{(1)}V_{2n+2}$$

(corresponding to (a) (b) (c))

The isomorphism is given by:

$$\Theta: \gamma_{2(n-k)} \oplus \gamma'_{2(n-k')} \oplus \gamma''_{2n+2} \longmapsto \gamma_{2(n-k)} D^{2k+2}/\!\sim \; + \gamma'_{2(n-k')} D^{2k'+2}/\!\sim \; + \gamma''_{2n+2}$$

<u>5.2 ${}^{(1)}V^*$ as a U_*-module</u>

We shall adapt Conner & Floyd's results (10) on the structure of $U_*(BZ/p)$ as a U_*-module. Milnor (see (10)) shows that for each odd prime p there exist polynomial generators x_{2p^k-2} of U^* having all Chern numbers divisible by p. First we examine these.

<u>Theorem 5.2.1</u> Let I(p) denote the ideal of U of all manifolds with all Chern numbers divisible by p . Then I(p) is generated by $x_0 = p$ points & x_{2p^k-2} $(k = 1,2,\ldots)$

<u>Proof</u> (Conner & Floyd (10) § 41)

There are various properties of I(p) which should have geometrical significance for the theories ${}^{(n)}V^*$:

<u>Theorem 5.2.2</u> I(p)= ideal of U^* of those bordism classes admitting a representative with a Z/p-action having a trivial normal bundle to the fixed point set. ("trivial normal bundle" means a sum of (trivial vector bundle)⊗(1 dim. representation of Z/p))

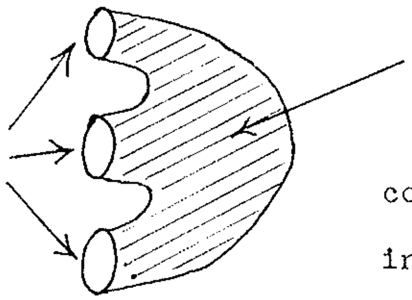
<u>Proof</u> Conner & Floyd (10) § 42.

<u>Theorem 5.2.3</u> Let I_n be the ideal of U^* generated by $x_0, x_{2p-2}, \ldots x_{2p^{n-1}-2}$. Denote by $SF((Z/p)^n)$ the ideal of $U^*$of those classes admitting representatives with fixed-point free $(Z/p)^n$-actions. Then $SF((Z/p)^n) = I_n$.

<u>Proof</u> Floyd (18) or tom Dieck (14).

Conner & Floyd ((10) § 46) use 5.2.2 and related results to find the Ω_*-module structure of $\Omega_*(BZ/p)$. We use their methods to calculate ${}^{(1)}V^*$ as a U_*-module. We follow them by defining, inductively, manifolds $M^{2k}, k = 1, 2, \ldots$ each with a Z/p-action t as follows:

<u>Definition 5.2.4</u> M^2 is a manifold with Z/p-action t, having p fixed points, given by taking a cobordism of py_1 to zero in $U_1(BZ/p)$ and "filling in the p copies of D^2":-



$$M^2 = N \cup_{py_1} pD^2$$

The Z/p-action t on M^2 is defined to be that on N extended over the discs so as to have a single fixed point at the centre of each disc.

Now suppose $M^2, M^4, \ldots, M^{2k}$ have all been defined, each with a Z/p-action t. We show how to define M^{2k+2}. Consider $D^2 \times M^{2k}$ with the action τ_1 of Z/p: $\tau_1(x,y) = (\rho x, y)$

(ρ the natural action of Z/p on $D^2 \hookrightarrow C$)

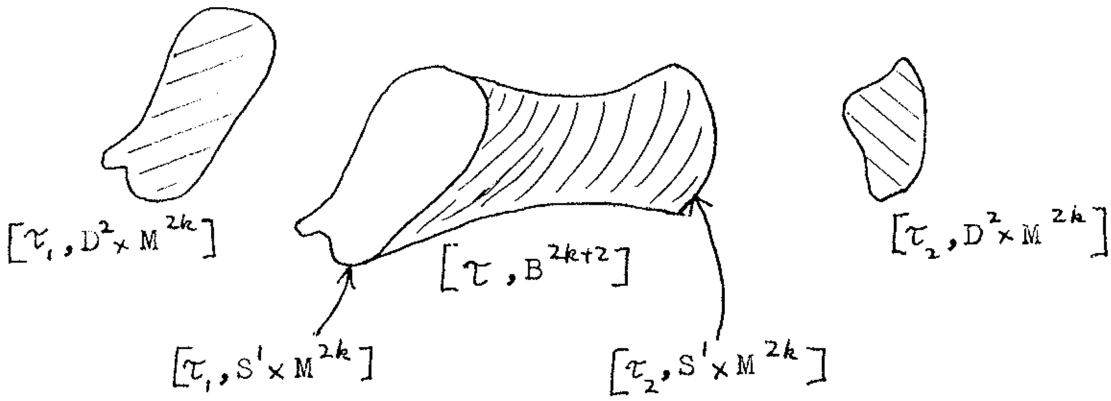
Also consider $D^2 \times M^{2k}$ with the Z/p-action τ_2: $\tau_2(x,y) = (\rho x, ty)$

Then τ_1 and τ_2 restricted to the boundary $S^1 \times M^{2k}$ give free actions of Z/p and hence define elements $[\tau_1, S^1 \times M^{2k}], [\tau_2, S^1 \times M^{2k}]$ in $U_{2k+1}(BZ/p)$.

<u>Lemma 5.2.5</u> (Conner & Floyd (10)) These are equal as elements of $U_{2k+1}(BZ/p)$.

<u>Proof</u> Conner & Floyd ((10) § 35.2) ▌

Let the cobordism of 5.2.5 be $[\tau, B^{2k+2}]$ (τ a free Z/p-action)



Definition 5.2.6 (Conner & Floyd (10))

M^{2k+2} is defined to be the union $(D^2 \times M^{2k}) \cup_{S^1 \times M^{2k}} B^{2k+2} \cup_{S^1 \times M^{2k}} (D^2 \times M^{2k})$ given by the picture above. The action t of Z/p on M^{2k+2} is defined to be that of τ_1, τ, τ_2 on the appropriate parts.

Proposition 5.2.7 (Conner & Floyd (10))

The fixed point set of t on M^{2k+2} is $[M^{2k}] \cup [-M^{2k-2}] \cup [M^{2k-4}] \cup \ldots$

Proof From the left end of the picture we have the fixed point set of $\tau_1 = M^{2k}$ (by definition of τ_1)

From the right end we have the fixed point set of τ_2 which is the fixed point set of $[t, M^{2k}]$

Thus the result follows by induction on k. ∎

Note that by its definition the action t has trivial normal bundle to its fixed point set (see 5.2.2 for what this means) and so by cutting out a neighbourhood of the fixed point set of t on M^{2k+2} Conner & Floyd interpret $[t, M^{2k+2}]$ as representing a cobordism:-

$$y_1[M^{2k}] - y_3[M^{2k-2}] + \ldots (-1)^k p y_{2k+1} = 0 \quad \text{in } U_{2k+1}(BZ/p) \quad \ldots\ldots (2)$$

(Kamata (22) deduces similar relations algebraically: let η be the Z/p-bundle $S^{2n+1} \longrightarrow S^{2n+1}/_{Z/p}$. The associated complex line bundle ζ has $\underbrace{\zeta \otimes \ldots \otimes \zeta}_{p}$=trivial bundle; thus $p.\log_{U*}(e_U(\zeta)) = 0$

where $\log_{\mathcal{U}^*}$ denotes the logarithm of the canonical formal group law on $U^*(-)$. Now Poincaré duality for $S^{2n+1}/_{Z/p}$ relates y_{2j+1} to $(e_u(\xi))^{n-j}$ and we know the coefficients of $\log_{\mathcal{U}^*}$ explicitly (Mishchenko see (8)), so $p.\log_{\mathcal{U}^*}(e_u(\xi)) = 0$ gives a series of relations by taking different n. These are the same as the relations (2).)

Just as we did in 5.1.2 for $CP^{p-1} \times I$, we examine $M^{2k+2} \times I$ with the action t and interpret it as a cobordism:

$$D^2/\sim.[M^{2k}] - D^4/\sim.[M^{2k-2}] + \ldots + (-1)^k p.D^{2k+2}/\sim \; = [M^{2k+2}] \text{ in } {}^{(1)}V^{2k+2} \ldots\ldots (3)$$

Conner & Floyd (10) show that the relations (2) generate the relations of $U_*(BZ/p)$ and also that the elements $[M^{2k+2}]$ and p generate I(p) as an ideal of U^*. (Their proof is for $\Omega_*(BZ/p)$ but applies equally for $U_*(BZ/p)$.) We deduce:

<u>Theorem 5.2.8</u> Let $z_{2k} = [M^{2k}] - D^2/\sim.[M^{2k-2}] + \ldots + (-1)^k p.D^{2k}/\sim$ Then ${}^{(1)}V^*$ is isomorphic to the free U^*-module generated by $1, D^2/\sim, D^4/\sim, \ldots, D^{2k}/\sim, \ldots$ quotiented by the (free) U^*-sub--module generated by $z_0 = p, z_2, z_4, \ldots, z_{2k}, \ldots$

<u>Proof</u> By (3) all the relations are necessary. Suppose there were another relation, say:

$$D^2/\sim.[N^{2k}] + D^4/\sim[N^{2k-2}] + \ldots + D^{2k+2}/\sim.[N^2] = [N^{2k+2}] \ldots\ldots (4)$$

Then, under the map $j: {}^{(1)}V^* \longrightarrow U_{*-1}(BZ/p)$ (from the s.e.s. (1)) this gives:

$$y_1[N^{2k}] - y_3[N^{2k-2}] + \ldots + y_{2k-3}[N^2] = 0 \ldots\ldots (5)$$

But all the relations in $U_*(BZ/p)$ are generated by the relations (3). Thus (5) must have left hand side linearly dependant on the l.h.s.'s of (2).

So (4) has l.h.s. dependant on the l.h.s.'s of (3).

Therefore the only new relations introduced by (4) relate elements on the r.h.s., i.e. in image $(U^* \longrightarrow {}^{(1)}V^*)$, but all such are generated by z_0.