A Z/p Analogue for Unoriented Bordism

Shaun Bullett

Ph.D. Thesis

University of Warwick

November 1973

I should like to thank the Science Research Council for their financial support during this work, the Mathematics Institute at Warwick for providing such a stimulating mathematical environment, and above all my research supervisor Dr. Alan Robinson for his constant encouragement (and for not believing me on the frequent occasions my ideas were wrong).

Summary

$N^*(-) \xrightarrow{\mu} H^*(-;Z/2)$ (Thom). I set out to construct a cobordism theory $V^*(-)$, as closely analogous as possible to $N^*(-)$, mapping onto $H^*(-;Z/p)$ where p is an odd prime. We first recall the following properties of $N^*(-)$:-

$N^*(BZ/2) \cong N^*[[X]]$, where we may choose X to be $e_N(\xi)$ (the N-Euler class of the universal Z/2-bundle). Multiplication $Z/2 \times Z/2 \longrightarrow Z/2$ then induces a map $N^*[[X]] \longrightarrow N^*[[Y,Z]]$, $X \mapsto F_N(Y,Z)$ and Boardman, Quillen and others showed that (N^*, F_N) is iso--morphic to the universal "Z/2 formal group".

I consider ring theories $h^*(-)$ mapping onto $H^*(-;Z/p)$. Such theories have $h^*(BZ/p) \cong h^*[[\beta_h]] \otimes E[\alpha_h]$. (E denotes an exterior algebra). Corresponding to $m: Z/p \times Z/p \longrightarrow Z/p$ I define a "Z/p formal group law" in a pair of variables $\binom{\alpha_h}{\beta_h}$ and calculate algebraically the universal Z/p formal group in this situation. I then set up a certain cobordism ring ${}^{(\infty)}V^*(-)$ of U-manifolds with singularities of type Z/p, $Z/p * Z/p$, ..., $\underbrace{Z/p * \ldots * Z/p}_{n \text{ fold join}}$, ... with structure groups $Z/p \wr \Sigma_n$ and show that a natural subtheory $V^*(-) \hookrightarrow {}^{(\infty)}V^*(-)$ has $V^*(X) \twoheadrightarrow H^*(X;Z/p)$ for all X and $V^*(BZ/p)$ canonically isomorphic to the universal Z/p formal group (by defining a pair of "Euler classes" $\binom{\alpha_V}{\beta_V} \in \binom{V^1(BZ/p)}{V^2(BZ/p)}$ for Z/p-bundles). The method can be used to generate other cobordism ring theories.

Contents

1. Introduction

Throughout this work all manifolds are C^∞ and all group actions are smooth.

Let X be a CW complex and let $N^*(-)$ denote the cobordism theory dual to the bordism theory of unoriented manifolds, $N_*(-)$. Thom (41) showed that the natural map $\mu: N^*(X) \to H^*(X;Z/2)$ is an epimorphism of rings.

Later, using geometric Steenrod operations in $N^*(-)$, Quillen (33) and tom Dieck (6) proved that $N^*(-)$ is characterised by the algebraic property that $N^*(BZ/2)$ is canonically isomorphic to the "universal Z/2-formal group". A consequence is that there is a natural ring isomorphism $N^*(X) \cong N^* \otimes H^*(X;Z/2)$ and a canonical splitting of μ as a map of rings, both first observed by Boardman (4) (see Quillen (33)). In Chapter 2 all these results are examined in detail to motivate analogous Z/p results later.

The object of this thesis is to consider a corresponding problem for Z/p (for primes $p \neq 2$), in particular to find a multiplicative cobordism theory $V^*(-)$ mapping onto $H^*(-;Z/p)$ such that $V^*(BZ/p)$ is canonically isomorphic to the "universal Z/p-formal group". As $H^*(BZ/p;Z/p) \cong P[\beta] \otimes E[\alpha]$ (P denotes poly. E ext. algebras over Z/p), in Chapter 3 I define a notion of "Z/p-formal group" involving both 'α' and 'β' variables; then, using methods based on those of Lazard (25) and Fröhlich (19), I calculate algebraically the "universal Z/p-formal group" and its ground ring, which I denote A_V.

The next stage is to build a geometric theory $V^*(-)$ such that $V^*(BZ/p)$ is canonically isomorphic to this universal formal group. The theory must have a natural geometric pair

$\begin{pmatrix}\alpha_V\\ \beta_V\end{pmatrix} \in \begin{pmatrix}V^1(BZ/p)\\ V^2(BZ/p)\end{pmatrix}$ and as a first step, in Chapter 4, we set up a geometric theory ${}^{(1)}V^*(-)$ of U-manifolds with Z/p-singularities having such a natural pair of Euler classes $\begin{pmatrix}\alpha_V\\ \beta_V\end{pmatrix}$ for each Z/p--bundle. The requirements of a commutative multiplication lead us to define a series of cobordism groups ${}^{(n)}V^*(-)$, with multiplications ${}^{(m)}V^*(-)\times{}^{(n)}V^*(-)\to{}^{(m+n)}V^*(-)$, and finally a multiplicative theory ${}^{(\infty)}V^*(-)$. The elements of ${}^{(n)}V^*$ may be pictured as U-manifolds with singularities, where the links of points are S^{*-1}, $S^{*-2}*Z/p$, $S^{*-3}*Z/p*Z/p, \ldots,$ $S^{*-n-1}*\underbrace{Z/p*\ldots*Z/p}_{n\text{-fold join}}$ and the singularity strata have structure groups $Z/p\wr\Sigma_0$, $Z/p\wr\Sigma_1$, $Z/p\wr\Sigma_2, \ldots, Z/p\wr\Sigma_n$ on their normal bundles. Alternatively, by "cutting along the singularities" one may picture elements of ${}^{(n)}V^*$ as U-manifolds with faces and corners of index $\leq n$, having free Z/p-actions on each face, the actions being "orthogonal" at corners, in the sense that on a corner of index r they combine locally to give a free action of $(Z/p)^r$ and hence globally to give an action of $Z/p\wr\Sigma_r$. (Σ_r carries the global "labelling data" of the faces).

As might be expected, the theories most amenable to calculation are ${}^{(1)}V^*(-)$ and ${}^{(\infty)}V^*(-)$. In Chapter 5, using techniques derived from those of Conner and Floyd (10) I calculate ${}^{(1)}V^*$ as an abelian group and as a U^*-module. (A consequence of this calculation, not proved here, is that there exist CW complexes X with ${}^{(1)}V^*(X)\to H^*(X;Z/p)$ not epimorphic, in contradiction to a conjecture of Sullivan (40).). However the theory we are really interested in is ${}^{(\infty)}V^*(-)$; $\begin{pmatrix}\alpha_V\\ \beta_V\end{pmatrix}\in\begin{pmatrix}{}^{(\infty)}V^*(BZ/p)\\ {}^{(\infty)}V^*(BZ/p)\end{pmatrix}$ induce a Z/p--formal group structure on ${}^{(\infty)}V^*(BZ/p)$ and thus we have a canonical map $\phi: A_V \to {}^{(\infty)}V^*$. In Chapter 6 I show that ϕ is injective, i.e. that ${}^{(\infty)}V^*(BZ/p)$ "carries" the universal Z/p-

-formal group law, and that as a consequence we may geometrically realise the Z/p-analogue $V^*(-)$ of $N^*(-)$ as a naturally included "sub-theory" of ${}^{(\infty)}V^*(-)$.

The reason that $V^*(-)$ has to be constructed indirectly, via ${}^{(\infty)}V^*(-)$, is that while $Z/\underbrace{2*\ldots*Z/2}_{n\text{ fold join}}$ is isomorphic to S^{n-1} and thus is homogeneous, so that we may extend the structure group $Z/2 \wr \Sigma_n$ to $O(n)$, $Z/\underbrace{p*\ldots*Z/p}_{n\text{ fold join}}$ has no such homogeneity. I conjecture that ${}^{(\infty)}V^*(-)$ is to $V^*(-)$ (multiplicatively) as $\{-,QCP^\infty\}$ is to $\{-,BU\}$ (additively), in the following sense: Segal (36) characterises $\{-,QCP^\infty\}$ (Q denotes $\Omega^\infty S^\infty$) as the minimal representable functor containing the group of formal sums of line bundles. He shows that BU is a direct factor of QCP^∞, so that $\{-,BU\}$ is a split summand of $\{-,QCP^\infty\}$. I conjecture that ${}^{(\infty)}V^*(-)$ is the universal theory containing the commutative group of formal products of Z/p-Euler classes, in some appropriate sense, and that $V^*(-)$ should be a split subring of ${}^{(\infty)}V^*(-)$.

In Chapter 7 I sketch briefly the construction of operations in ${}^{(\infty)}V^*(-)$ and $V^*(-)$ and outline some further possible lines of development. The method of introducing join-type singularities with wreath product structure groups appears to be a useful method of generating ring theories.

For convenience, cohomology theories $h^*(-)$ will mostly be pictured here using Quillen's notion (33) of cobordism of h-oriented maps of manifolds or else in mock-bundle terms. For general results on mock-bundles I refer to Rourke and Sanderson (35 for manifolds with singularities see Sullivan (40), Baas (3), Stone (38) and for manifolds with corners see Douady (16); for algebraic results on formal groups see Fröhlich (19) and for applications in algebraic topology see Quillen (33), tom Dieck (6), Bukhštaber Mishchenko and Novikov (8), Adams (1) etc.

2. $N^*(-)$ and Z/2-formal groups

We examine Quillen's (33), Bröcker & tom Dieck's (6), and Boardman's results about $N^*(-)$ and the universal Z/2--formal group, as a preliminary to our construction of a Z/p analogue.

2.1 Z/2-formal groups

Let B be a finite CW complex and $\eta: E \to B$ be a real n-dim. vector bundle.

Definition 2.1.1 The Thom space $T(\eta)$ of η is the disc bundle $D(\eta)$ of η, with its boundary sphere bundle $S(\eta)$ identified to a point.

For each η there is a canonical class $t_N(\eta) \in \tilde{N}^n(T(\eta))$ called the Thom class of η (see Bröcker & tom Dieck (6)).

Remark 2.1.2 For B a manifold we may use Quillen's description of $N^*(B)$ as cobordism classes of maps of manifolds $f: M \to B$ carrying certain normal structure ((33) or see 4.1) and then the manifold B embedded as the zero section in $D(\eta)$ represents the class $t_N(\eta) \in N^n(D(\eta), S(\eta)) \cong \tilde{N}^n(T(\eta))$.

Definition 2.1.3 Let i be the inclusion of the zero section $i: B^+ \hookrightarrow T(\eta)$ (B^+ denotes B with a disjoint base point added) The N-Euler class $e_N(\eta)$ of η is defined to be $e_N(\eta) = i^* t_N(\eta)$ in $N^n(B)$.

For the canonical line bundle ξ_n over RP^n, $T(\xi_n) \cong RP^{n+1}$ and $t_N(\xi_n)$ may be represented by the embedded submanifold $RP^n \hookrightarrow RP^{n+1}$. Thus $e_N(\xi_n) \in N^1(RP^n)$ may be represented by $RP^{n-1} \hookrightarrow RP^n$.

Proposition 2.1.4 Let $i: RP^n \hookrightarrow RP^{n+1}$.

Then (i) $i^* e_N(\xi_{n+1}) = e_N(\xi_n)$

so $\varprojlim_n e_N(\xi_n)$ defines a unique element $e_N(\xi) \in \varprojlim N^1(RP^n$

(ii) $N^*(RP^n) \cong N^*[X]/X^{n+1}$ where $X = e_N(\xi_n)$

so $N^*(RP^{n+1}) \xrightarrow[i^*]{} N^*(RP^n)$ and as a result we have

$N^*(RP^\infty) \cong \varprojlim_n N^*(RP^n)$ (all i^* epi $\Rightarrow \varprojlim_n{}^1 (N^*(RP^n)) = 0$

(see Bröcker & tom Dieck (6))

Proof Bröcker & tom Dieck (6) ▌

Corollary 2.1.5 $N^*(RP^\infty) \cong N^*[[X]]$ (formal power series ring)

where $X = e_N(\xi)$ and $N^*(RP^\infty)$ is topologised by the skeletal filtration $RP^n \hookrightarrow RP^{n+1} \hookrightarrow \ldots$ of RP^∞; $N^*[[X]]$ is topologised by the ideals generated by $X, X^2, X^3, \ldots$

Also $N^*(RP^\infty \times RP^\infty) \cong N^*[[Y,Z]]$

where $Y = e_N(\xi')$; $Z = e_N(\xi'')$ (ξ' pull-back of ξ under projection onto 1st factor $RP^\infty \times RP^\infty \xrightarrow[\pi_1]{} RP^\infty$, ξ'' similarly for 2nd factor)

Proof Bröcker & tom Dieck (6) ▌

$BZ/2 \simeq RP^\infty$ (RP^∞ classifies line bundles and thus $Z/2$-bundles)

Multiplication $m: Z/2 \times Z/2 \to Z/2$ is a group homomorphism, as $Z/2$ is abelian, and so m induces:

$m: B(Z/2 \times Z/2) \simeq (BZ/2 \times BZ/2) \to BZ/2$ (thinking of $BZ/2$ as RP^∞ this classifies the $\otimes$-bundle over $RP^\infty \times RP^\infty$)

$m^*: N^*[[X]] \to N^*[[Y,Z]]$ is an N^*-algebra morphism sending $1 \in N^*$ to $1 \in N^*$ and so m^* is determined by its value on X.

$$m^*: X \mapsto F_N(Y,Z) = Y + Z + \sum_{\substack{i \geq 1 \\ j \geq 1}} a_{ij} Y^i Z^j \quad (a_{ij} \in N^*)$$

F satisfies certain properties from the group structure of $Z/2$:-

(1) $F(0,X) = F(X,0) = X$ -identity

(2) $F(X,F(Y,Z)) = F(F(X,Y),Z)$ -associativity

(3) $F(X,Y) = F(Y,X)$ -commutativity

(4) $F(X,X) = 0$ -order 2

Definition 2.1.6 for R a commutative algebra over $Z/2$, call an $F \in R[[X,Y]]$ satisfying (1),(2),(3) & (4) a Z/2-formal group law over R.

Among pairs (R,F) there is a universal pair (L,F_L) such that for each (R,F) there is a unique algebra homomorphism $\phi:L\to R$ sending each coefficient l_{ij} of F_L to the corresponding coefficient a_{ij} of F.

(It is clear that such an L exists-it is a quotient of the algebra $Z/2[a_{ij};\text{all } i,j>0]$ by the relations of (1),(2),(3),(4))

<u>Definition 2.1.7</u> A <u>logarithm for F</u> is a series $l_F(X) = X+a_1X^2+ +a_2X^3+\ldots\in R[[X]]$ such that $l_F(F(X,Y)) = l_F(X)+l_F(Y)$

<u>Theorem 2.1.8 (Lazard)</u>

Every Z/2-formal group law has a logarithm. This logarithm may be chosen with $a_i=0$ for all $i=2^j-1$ and is unique in this form. We therefore have:

$$L\cong Z/2[l_i;\ i\neq 2^j-1]$$

$$l_{F_L}=X+l_2X^3+l_4X^5+l_5X^6+\ldots$$

and so $F_L(X,Y) = l_{F_L}^{-1}(l_{F_L}(X)+l_{F_L}(Y))$ (l^{-1} means inverse series i.e. $l^{-1}(l(X))=X$.)

The unique $\phi:L\to R$ sending F_L to F is generated by sending the coefficients of l_{F_L} to the corresponding coefficients of the canonical l_F.

<u>Proof</u> Fröhlich (19) or see 3.5.4, which applies for $p=2$. ∎

<u>Theorem 2.1.9 (Quillen & tom Dieck)</u>

$N^*\cong L$. The formal group law F_N is isomorphic to F_L.

<u>Proof</u>

(An outline of a proof only will be given here; it will be in the spirit of Boardman's work (4) and all details are explicit or implicit there.)

By the theorem of Thom (41) (or see Rourke (34)), the Thom spectrum $\underset{\sim}{MO}$ for $N^*(-)$ is equivalent to a product of copies of $\underset{\sim}{K(Z/2)}$ (Eilenberg-Maclane spectra).

Thus we have a canonical ring isomorphism:

$$N^*(X) \cong \{X, \underset{\sim}{MO}\} \cong \mathrm{Hom}_{\mathcal{A}_2}[H^*(MO;Z/2), H^*(X;Z/2)]$$

where $\mathcal{A}_2$ denotes the Z/2-Steenrod algebra, and where the ring structure on the last group is made from cup product on $H^*(X;Z/2)$ together with the comultiplication on $H^*(\underset{\sim}{MO};Z/2)$ induced by the product map of spectra $\underset{\sim}{MO} \wedge \underset{\sim}{MO} \longrightarrow \underset{\sim}{MO}$.

So $N^*(X) \underset{B}{\hookrightarrow} \mathrm{Hom}_{Z/2}[H^*(\underset{\sim}{MO};Z/2), H^*(X;Z/2)] \cong H^*(X;Z/2) \hat{\otimes} H_*(\underset{\sim}{MO};Z/2)$.

B is called the <u>Boardman map</u>. ($\hat{}$ denotes completion with respect to neighbourhoods $\bigoplus_{*<n} H^*(X) \otimes H_*(\underset{\sim}{MO})$; see Boardman (4) for details of $\hat{\otimes}$). To prove 2.1.9 we shall need the following:-

<u>Lemma 2.1.10</u> (Boardman) There is an algebra isomorphism $H_*(\underset{\sim}{MO};Z/2) \cong Z/2[a_1, a_2, \ldots]$ under which the Boardman map sends:

$$\begin{array}{ccc} e_N(\xi) & \longmapsto & f(t) = t + a_1 t^2 + a_2 t^3 + \ldots \\ \cap & & \cap \\ N^1(BO(1)) & & H^{*+1}(BO(1);Z/2) \hat{\otimes} Z/2[a_1, a_2, \ldots] \end{array}$$

<u>Proof of lemma</u>

$BO(1) = BZ/2$ so $H^*(BO(1);Z/2) \cong Z/2[t]$

$\underbrace{BO(1) \times \ldots \times BO(1)}_{n} \longrightarrow BO(n)$ induces a "splitting map":

$$H^*(BO(n);Z/2) \hookrightarrow H^*(\underbrace{BO(1) \times \ldots \times BO(1)}_{n};Z/2)$$ with image all symmetric functions in $t_1, \ldots, t_n$. This injection sends:

Stiefel-Whitney class $w_i \longmapsto$ i th elem. sym. fn. in $t_1, \ldots, t_n$

The maps $BO(m) \times BO(n) \longrightarrow BO(m+n)$ "adding bundles", correspond under the splitting to:

$$\begin{array}{ccccc} H^*(BO(m+n);Z/2) & \xrightarrow{m^*} & H^*(BO(m);Z/2) & \otimes & H^*(BO(n);Z/2) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(\underbrace{BO(1) \times \ldots \times BO(1)}_{m+n};Z/2) & \longrightarrow & H^*(\underbrace{BO(1) \times \ldots \times BO(1)}_{m}) & \otimes & H^*(\underbrace{BO(1) \times \ldots \times BO(1)}_{n}) \end{array}$$

$$t_i \longmapsto t_i' \otimes 1 \quad (i \leq n)$$

$$t_i \longmapsto 1 \otimes t''_{i-n} \quad (i > n) \qquad \ldots\ldots\ldots\ldots (1)$$

From the Stiefel-Whitney classes w_i we obtain a basis for $H_*(BO;Z/2)$ as follows:

The evaluation map $e: H^*(BO(n);Z/2) \otimes H_*(BO(n);Z/2) \to Z/2$ has dual $e': Z/2 \to H_*(BO(n);Z/2) \hat{\otimes} H^*(BO(n);Z/2)$

The image of 1 under the dual to the evaluation map is called a <u>universal element</u> by Boardman (see (4) for details of this approach to duality).

Consider the element

$$u_n = (1 + a_1 t_1 + a_2 t_1^2 + \dots)\dots(1 + a_1 t_n + a_2 t_n^2 \dots)$$
$$\in H^*(\underbrace{BO(1) \times \dots \times BO(1)}_{n};Z/2) \hat{\otimes} Z/2[a_1, a_2, a_3, \dots]$$

(a_i variable in dim. i)

u_n may be viewed as a universal element giving an isomorphism of some subspace A_n of $Z/2[a_1, a_2, \dots]$ to the dual B_n' of a subspace B_n of $H^*(\underbrace{BO(1) \times \dots \times BO(1)}_{n};Z/2)$.

$$u_n = 1 + a_1(t_1 + \dots t_n) + a_2(t_1^2 + \dots t_n^2) + a_1^2(t_1 t_2 + t_1 t_3 + \dots + t_{n-1} t_n) + \dots$$

It is clear that each polynomial in the a_i's of total degree $\leq n$ is represented in u_n with a different non-zero coefficient. Thus $A_n \cong$ subspace of $Z/2[a_1, a_2, \dots]$ of polynomials of degree $\leq n$. Also each coefficient is a symmetric polynomial in $t_1, \dots, t_n$, so that $B_n \hookrightarrow H^*(BO(n);Z/2) \hookrightarrow H^*(\underbrace{BO(1) \dots BO}_{n}(1);Z/2)$. But A_n is the same size as $H_*(BO(n);Z/2)$ in each dimension, so $B_n \cong H^*(BO(n);Z/2)$, and u_n gives an isomorphism $A_n \cong H_*(BO(n);Z/2)$

In the limit $n \to \infty$, $BO(n) \to BO$, and u gives an isomorphism $Z/2[a_1, a_2, \dots] \cong H_*(BO;Z/2)$; further, from the form of $u = (1 + a_1 t_1 + a_2 t_1^2 + \dots)(1 + a_1 t_2 + \dots)\dots$, this is an isomorphism of algebras (where the algebra structure on $H_*(BO;Z/2)$ is dual to the coalgebra structure on $H^*(BO;Z/2)$, given by 1).

The Thom isomorphism $H^*(BO(n);Z/2) \cong H^{*+n}(MO(n);Z/2)$ is multiplication by $t_1 t_2 \dots t_n$. Applying this to the element u_n we get an element:

$$u'_n = (t_1 + a_1 t_1^2 + a_2 t_1^3 + \ldots)(t_2 + a_1 t_2^2 + \ldots)\ldots(t_n + a_1 t_n^2 + \ldots)$$

$$\in H^{*+n}(MO(n);Z/2)\hat{\otimes} Z/2[a_1, a_2, a_3, \ldots]$$

$$(H^{*+n}(MO(n)) \hookrightarrow H^{*+n}(BO(n)) \hookrightarrow H^{*+n}(\underbrace{BO(1)\times\ldots\times BO(1)}_{n}))$$

In the limit as $n\to\infty$ u' gives an algebra isomorphism:

$Z/2[a_1, a_2, a_3, \ldots] \cong H_*(\underset{\sim}{MO};Z/2)$ $(\equiv \varinjlim H_{*+n}(MO(n);Z/2))$.

(The multiplication on $H_*(\underset{\sim}{MO};Z/2)$ comes from $\underset{\sim}{MO}\wedge\underset{\sim}{MO} \longrightarrow \underset{\sim}{MO}$, which in turn comes from the maps $BO(m)\times BO(n) \longrightarrow BO(m+n)$.)

The identity map $\in \{\underset{\sim}{MO}, \underset{\sim}{MO}\}$ gives an element $\iota \in N^0(\underset{\sim}{MO})$ $(\equiv \varprojlim N^n(MO(n)))$. Under the Boardman map $B: N^*(\underset{\sim}{MO}) \hookrightarrow H^*(\underset{\sim}{MO};Z/2)\hat{\otimes} H_*(\underset{\sim}{MO};Z/2)$ ι maps to the identity map in $\mathrm{Hom}_{Z/2}[H^*(\underset{\sim}{MO};Z/2), H^*(\underset{\sim}{MO};Z/2)]$ i.e. to the universal element in $H^*(\underset{\sim}{MO};Z/2)\hat{\otimes} H_*(\underset{\sim}{MO};Z/2)$, which in terms of our bases is u'.

The identity map $\in \{MO(n), MO(n)\}$ gives rise to an element $\iota_n \in N^n(MO(n))$; $(\iota = \varprojlim \iota_n)$. Under the Boardman map $B: N^*(MO(n)) \hookrightarrow H^*(MO(n);Z/2)\hat{\otimes} H_*(\underset{\sim}{MO};Z/2)$ ι_n maps to the restriction of u', i.e. ι_n maps to the element $u'_n \in H^{*+n}(MO(n))\hat{\otimes} Z/2[a_1, a_2, \ldots]$

Thus, in particular, ι_1 represents an element of $N^1(MO(1))$ which maps to : $t + a_1 t^2 + a_2 t^3 + \ldots \in H^*(MO(1);Z/2)\hat{\otimes} Z/2[a_1, a_2, \ldots]$

But ι_1 is just the Thom class $\in N^1(T(\xi))$ of the canonical line bundle ξ over $BO(1)$ so $i^*\iota_1 = e_N(\xi)$ (where $i: BO(1) \hookrightarrow MO(1)$). Thus, by the naturality of B, we have:

$$\begin{array}{ccc} e_N(\xi) & \xmapsto{B} & f(t) = t + a_1 t^2 + a_2 t^3 + \ldots \\ \cap & & \cap \\ N^1(BO(1)) & & H^{*+1}(BO(1);Z/2)\hat{\otimes} Z/2[a_1, a_2, a_3, \ldots] \end{array}$$

completing the proof of the lemma 2.1.10. ▌

We now return to the proof of the theorem 2.1.9:-

Let m be the multiplication map $BO(1)\times BO(1)\to BO(1)$ $(BO(1) = BZ/2)$

Then $m^*: H^*(BO(1);Z/2)\to H^*(BO(1)\times BO(1);Z/2)$

sends $t \longmapsto t'+t''$

$m^*: N^*(BO(1)) \longrightarrow N^*(BO(1)\times BO(1))$

sends $e_N(\xi) \longmapsto F_N(e_N(\xi'),e_N(\xi''))$

Thus using m^* and the naturality of B,

$$B: F_N(e_N(\xi'),e_N(\xi'')) \longmapsto f(t'+t'')$$

Write $X=e_N(\xi')$, $Y=e_N(\xi'')$, $X_1=$ image X $(=f(t'))$, $Y_1=f(t'')$.

$$\text{Then}\quad \underset{N^*[[X,Y]]}{\underset{\cap}{B:F_N(X,Y)}} \longmapsto \underset{Z/2[a_1,a_2,\ldots][[X_1,Y_1]]}{\underset{\cap}{f\left[f^{-1}(X_1)+f^{-1}(Y_1)\right]}}$$

Let $f^{-1}(X)=X+b_1X^2+b_2X^3+\ldots$ be the series formally inverse to f, so that

$$\begin{cases} b_1 = a_1 + \text{composite terms} \\ b_2 = a_2 + \ldots \\ \ldots\ldots \end{cases}$$
i.e. b_i are poly. gens for $Z/2[a_1,a_2,\ldots]$

Consider the map $Z/2[a_1,a_2,\ldots]\xrightarrow{\Phi} L$

given by

$$\begin{cases} b_{2^j-1} \longmapsto 0 \\ b_i \longmapsto l_i \quad \text{otherwise}\end{cases}$$

Then $\Phi B: N^*\to Z/2[a_1,a_2,\ldots]\longrightarrow L$

$$\text{sends}\quad \underset{N^*[[X,Y]]}{\underset{\cap}{F_N(X,Y)}} \longmapsto \underset{L[[X_1,Y_1]]}{\underset{\cap}{l^{-1}(l(X_1)+l(Y_1))=F_L(X_1,Y_1)}}$$

So $\Phi B: N^*\to L$ splits the canonical $\phi: L\to N^*$ (recall from 2.1.8 that ϕ is the unique ring map sending F_L to F_N). Thus ϕ is injective; to complete the theorem it only remains to show ϕ is epimorphic and we do this by showing N^* and L are the same size in each dimension:

$$H^*(\underset{\sim}{MO};Z/2) \text{ free over } \mathcal{A}_2 \Rightarrow H_*(\underset{\sim}{MO};Z/2)\cong \pi_*(\underset{\sim}{MO})\otimes \mathcal{S}_2 \cong N^*\otimes \mathcal{S}_2$$

(where $\mathcal{S}_2$ is the dual to $\mathcal{A}_2$; Milnor (27) shows $\mathcal{S}_2\cong Z/2[\lambda_1,\lambda_2,\ldots]$, λ_i in dimension 2^i-1).

But $H_*(\underset{\sim}{MO};Z/2)\cong Z/2[a_1,a_2,a_3,\ldots]$ so N^* is the same size as $L\cong Z/2[l_2,l_4,\ldots,l_i,\ldots]$ $(i\neq 2^j-1)$. ∎

2.2 The ring splitting $\nu: H^*(-;Z/2) \to N^*(-)$

Proposition 2.2.1 Any stable multiplicative natural transformation $\nu: H^*(-;Z/2) \to N^*(-)$ is determined by its value on the space $BZ/2$.

Proof Let $\nu: H^*(BZ/2;Z/2) \to N^*(BZ/2)$

send $t \longmapsto f(e_N(\xi))$ (f a power series over N^*)

Since ν is multiplicative f determines ν on $H^*(BZ/2;Z/2) \cong Z/2[t]$ Let K^{2n} be the 2n-skeleton of the Eilenberg-Maclane space $K(Z/2,n)$. Then:

$H^*(K^{2n};Z/2) \hookrightarrow H^*(\underbrace{BZ/2 \times \ldots \times BZ/2}_{n};Z/2)$ (this injection is induced by the map classifying the Z/2-cohomology class $t_1 t_2 \ldots t_n \in H^*(\underbrace{BZ/2 \times \ldots \times BZ/2}_{n};Z/2)$; see Steenrod & Epstein (37))

Since the Atiyah-Hirzebruch spectral sequence $H^*(-;N^*) \Rightarrow N^*(-)$ collapses we have the same inclusion:

$N^*(K^{2n}) \hookrightarrow N^*(\underbrace{BZ/2 \times \ldots \times BZ/2}_{n})$

Thus, by multiplicativity and naturality, ν is determined on $H^*(K^{2n};Z/2)$ and so on the first n cohomology groups of all complexes of dimension $<2n$. But ν is stable so this determines it on all finite dimensional complexes; however $H^*(X;Z/2) \cong \cong \varprojlim H^*(X_n;Z/2)$ for X CW and X_n n-skeleton of X (see Bröcker & tom Dieck (6)) and as the spectral sequences collapse, $N^*(X) \cong \varprojlim N^*(X_n)$. Thus ν is determined on all CW X. ∎

Corollary 2.2.2 If $\nu: H^*(-;Z/2) \to N^*(-)$ is as in 2.2.1 , and $\nu: t \longmapsto f(e_N(\xi))$, then f is a logarithm of F_N.

Proof Let η be any Z/2-bundle over a CW X and let $w(\eta) \in H^1(X;Z/$ denote its 1st Stiefel-Whitney class. Then by the naturality of ν, $\nu: w(\eta) \longmapsto f(e_N(\eta)) \in N^1(X)$

In particular let $X=BZ/2\times BZ/2$ and $\eta=\zeta'\otimes\zeta''$.

Then $\nu: w(\zeta'\otimes\zeta'')\mapsto f(e_N(\zeta'\otimes\zeta''))\quad\in N^1(BZ/2\times BZ/2)$

But $w(\zeta'\otimes\zeta'') = t'+t''\in H^1(BZ/2\times BZ/2:Z/2)$ and ν preserves addition

so $\quad \nu: t'+t''\mapsto f(e_N(\zeta'))+f(e_N(\zeta''))$

i.e. $\quad f(e_N(\zeta'\otimes\zeta''))=f(e_N(\zeta'))+f(e_N(\zeta''))$ – f is a logarithm of F_N ∎

Proposition 2.2.3 The stable multiplicative natural transformations $\nu: H^*(-;Z/2)\longrightarrow N^*(-)$ are in 1-1 correspondance with the logarithms of F_N ; the canonical ν is given by the the canonical logarithm. (see 2.1.8)

Proof Let $h^*(-)=H^*(-;Z/2)\otimes R$ (R an algebra over Z/2) A sequence of elements $u_n\in h^n(MO(n))$ determines a stable natural transformation $N^*(-)\rightarrow h^*(-)$ if the map $SMO(n)\rightarrow MO(n+1)$ sends $u_{n+1}\mapsto u_n$. The transformation is multiplicative if the maps $MO(m)\times MO(n)\longrightarrow MO(m+n)$ send $u_{m+n}\mapsto u_m\times u_n$.

Put $u_1=t\otimes 1+t^2\otimes r_1+t^2\otimes r_2+\dots\in h^1(MO(1))=H^1(-;Z/2)\otimes R$ Then the stability and multiplicativity conditions above determine a unique sequence of elements u_n. (using the injections $h^*(MO(m+n))\hookrightarrow h^*(MO(m))\otimes h^*(MO(n))$ etc. We omit the details-they are straightforward)

Thus each series $t\otimes 1+t^2\otimes r_1+t^3\otimes r_2+\dots$ determines a unique stable multiplicative natural transformation $N^*(-)\rightarrow h^*(-)$. In particular, put R=L and let the series be $l^{-1}(t)$ for l any logarithm of F . We then get a natural transformation $N^*(-)\rightarrow H^*(-;Z/2)\otimes L$ which sends the formal group law F_N to the group law $l^{-1}(l(-)+l(-))$ i.e. to F_L. Thus the natural transformation is an isomorphism on the point rings and so gives an isomorphism on all X.

We have proved each logarithm determines a ring isomorphism $N^*(-)\cong H^*(-;Z/2)\otimes L$ and thus a ring splitting $\nu: H^*(-;Z/2)\rightarrow N^*(-)$; 2.2.1 and 2.2.2 complete the proposition. ∎

2.3 The Action of the Steenrod Algebra $\mathcal{A}_2$

We use Milnor's description (27) of the action of $\mathcal{A}_2$ on $H^*(BZ/2;Z/2)$. Let ζ_2 = dual Hopf algebra to $\mathcal{A}_2$.

Theorem 2.3.1 (Milnor)

$\zeta_2 \cong Z/2\,[\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots; \ \lambda_i \text{ dim. } 2^i-1]$

The action $H^*(BZ/2;Z/2)\otimes\mathcal{A}_2 \to H^*(BZ/2;Z/2)$ is described by giving the dual coaction:

$\theta: H^*(BZ/2;Z/2) \to H^*(BZ/2;Z/2)\otimes\zeta_2$

which is a ring homomorphism generated by:

$$t \longmapsto t\otimes 1 + t^2\otimes\lambda_1 + t^4\otimes\lambda_2 + t^8\otimes\lambda_3 + \ldots \qquad \ldots(1)$$

$(H^*(BZ/2;Z/2) \cong Z/2[t]\ )$

Proof Milnor (27) ▮

Remark 2.3.2 The formal group $H^*(BZ/2;Z/2) \cong Z/2[t]$ has law $F_H(t', t'') = t' + t''$ and ζ_2 gives "the most general coaction preserving F_H." To be precise: (ζ_2, θ) is universal among pairs (A,f) where A is a $Z/2$-algebra and $f: t \mapsto t\otimes 1 + t^2\otimes a_1 + t^3\otimes a_2 + \ldots$ is a ring homomorphism $H^*(BZ/2;Z/2) \to H^*(BZ/2;Z/2)\otimes A$ which preserves the formal group law F_H. (It is clear that if $f(t'+t'') = f(t') + f(t'')$ (mod 2) then f must be of form (1), so that (ζ_2, θ) is indeed universal as claimed.) Of course any coalgebra A corresponding to an algebra of additive operations on $H^*(-;Z/2)$ must preserve F_H, by naturality and the argument of 2.2.2, so this says $\mathcal{A}_2$ is "as large as it can be".

In fact any power series transformation preserving any $Z/2$-formal group law must be of form (1), so that for example (ζ_2, θ) is also universal among pairs (A,f) where A is a $Z/2$-algebra and $f: e_N(\xi) \mapsto e_N(\xi)\otimes 1 + (e_N(\xi))^2\otimes a_1 + \ldots$ is an N^*-algebra homomorphism preserving the $Z/2$-formal group law F_N. In this case $\mathcal{A}_2$ corresponds to the tom Dieck Steenrod operations on $N^*(-)$. (12)

When we have defined "Z/p-formal group laws" in Chapter 3 it will be clear that $\mathfrak{S}_p$ (the dual to $\mathcal{Q}_p$) is "the most general coaction preserving such a law" in just the same way.

3. Z/p-Formal Groups

3.1 Z/p-Theories

Definition 3.1.1 (Rourke (34)) A cohomology theory $h^*(-)$ is called a Z/p-theory if and only if it satisfies the following three conditions:

(1) It is connected. (i.e. $h_i(pt.) = 0$ for $i<0$)

(2) It is a ring theory.

(3) It has $h_0(pt.) \cong Z/p$.

(Note that (2) & (3) $\Rightarrow h^*(X)$ is a Z/p-algebra for all X.)

$h^*(-)$ is further said to be commutative (in the sense of Milnor (27)) if $a.b = (-1)^{\dim a.\dim b} b.a$

(e.g. $H^*(-;Z/p)$ is a commutative Z/p-theory; $N^*(-)$ and $H^*(-;Z/2)$ are commutative Z/2-theories.)

$H^*(BZ/p;Z/p) \cong P[\beta] \otimes E[\alpha]$ (see Steenrod and Epstein (37); we use P to denote polynomial, E exterior, algebras over Z/p; $\alpha \in H^1(BZ/p;Z/p)$ $\beta \in H^2(BZ/p;Z/p)$.)

Since we are looking for a Z/p-theory $V^*(-)$, universal in some sense among Z/p-theories mapping onto $H^*(-;Z/p)$, we examine theories $h^*(-)$ with $h^*(BZ/p) \cong h^* \otimes H^*(BZ/p;Z/p)$ i.e. with $h^*(BZ/p) \cong h^*[[\beta_h]] \otimes E[\alpha_h]$ (this may be regarded as a "free" power series ring in α_h, β_h, over h^*, as the commutativity condition ensures $\alpha_h^2 = 0$.)

In order to have a natural formal group structure defined on $h^*(BZ/p)$ we shall require $h^*(-)$ to be a theory having, for the universal Z/p-bundle ξ a "canonical pair of Z/p-Euler classes" $\begin{pmatrix} \alpha_h(\xi) \in h^1(BZ/p) \\ \beta_h(\xi) \in h^2(BZ/p) \end{pmatrix}$ (see Definition 3.3.1) analogous to $e_N(\xi) \in N^1(BZ/2)$ in the Z/2 case.

<u>Theorem 3.1.2 (Rourke)</u> For any Z/p-theory $h^*(-)$ the natural map $\mu: h^*(X) \to H^*(X;Z/p)$ is epimorphic for all X if and only if it is epimorphic $h^*(BZ/p) \to H^*(BZ/p;Z/p)$.

<u>Proof</u> Rourke (34) (The main ingredient is a splitting principle similar to that used in 2.2.1. This is that the map $\underbrace{BZ/p \times \ldots \times BZ/p}_{2n} \to K(Z/p,3n)$ classifying $\alpha\beta\alpha\ldots\beta$ gives an injection of the Z/p-cohomology of the 4n-skeleton of $K(Z/p,3n)$)

<u>Corollary 3.1.3 (Rourke)</u> If $h^*(-)$ is a Z/p- theory and there are classes $\alpha_h \in h^1(BZ/p)$, $\beta_h \in h^2(BZ/p)$ such that $\mu(\alpha_h) = \alpha \in H^1(BZ/p;Z/p)$, and $\mu(\beta_h) = \beta \in H^2(BZ/p;Z/p)$, then $\mu: h^*(X) \to H^*(X;Z/p)$ is epimorphic for all X.

<u>Proof</u> Immediate. ∎

This gives a very useful easy test for whether a Z/p--theory $h^*(-)$ maps onto $H^*(-;Z/p)$.

<u>3.2 The Action of the Steenrod Algebra $\mathcal{Q}_p (p \neq 2)$</u>

We use Milnor's description (27) of the action of $\mathcal{Q}_p$ on $H^*(BZ/p;Z/p)$. Let $\mathfrak{S}_p$ = dual Hopf algebra to $\mathcal{Q}_p$.

<u>Theorem 3.2.1 (Milnor)</u>

$\mathfrak{S}_p \cong P[\xi_1, \xi_2, \ldots; \xi_i \text{ dim. } 2p^i-2] \otimes E[\tau_0, \tau_1, \tau_2, \ldots; \tau_i \text{ dim. } 2p^i-1]$

The action $H^*(BZ/p;Z/p) \otimes \mathcal{Q}_p \to H^*(BZ/p;Z/p)$ is described by giving the dual coaction:

$$\theta: H^*(BZ/p;Z/p) \to H^*(BZ/p;Z/p) \otimes \mathfrak{S}_p$$

which is a ring homomorphism generated by:

$$\alpha \longmapsto \alpha\otimes 1 + \beta\otimes\tau_0 + \ldots + \beta^{p^i}\otimes\tau_i + \ldots$$
$$\beta \longmapsto \beta\otimes 1 + \beta^p\otimes\xi_1 + \ldots + \beta^{p^i}\otimes\xi_i + \ldots$$

<u>Proof</u> Milnor (27). ∎

<u>Remark 3.2.2</u> We shall define "Z/p-formal group law" in such a way that this is the most general coaction preserving one. (c.f. 2.3.2.)

3.3 The Inclusion $Z/p \hookrightarrow S^1$

To motivate a part of the definition of a "Z/p-formal group" we examine a special property of $H^*(-;Z/p)$:-

Take any group inclusion $Z/p \underset{i}{\hookrightarrow} S^1$ and regard it as standard from now on in this work.

$$H^*(BS^1;Z/p) \cong P[\delta] \qquad (\delta \in H^2(BS^1;Z/p))$$

$$i^*: H^*(BS^1;Z/p) \hookrightarrow H^*(BZ/p;Z/p)$$

$$\delta \longmapsto \beta \qquad \text{(for appropriate choice of } \delta)$$

Definition 3.3.1 Let $h^*(-)$ be a commutative Z/p-theory containing classes $\alpha_h \in h^1(BZ/p)$, $\beta_h \in h^2(BZ/p)$ satisfying the conditions of Corollary 3.1.3 Then if β_h is in the image of i^* we call the triple $(h^*(-), \alpha_h, \beta_h)$ a representative Z/p-theory [If $\beta_h = i^*\delta_h$ for $\delta_h \in h^2(BS^1)$ then $h^*(BS^1) \cong h^*[[\delta_h]]$. As $BS^1 \simeq CP^\infty$ choosing a particular polynomial generator δ_h of $h^*(BS^1)$ is equivalent to assigning to each complex line bundle a canonical "Euler class" δ_h].

3.4 Z/p-Formal Groups

We consider commutative (in the sense of 3.3.1) graded algebras A^* over Z/p. (with $A^n = 0$ for $n>0$.) Let α be a formal variable (graded 1) and β be a formal variable (graded 2).

As in the Z/2 case we examine graded A^*-algebra homomorphisms m^* sending 1 to 1 :-

$$m^*: A^*[[\beta]] \otimes E[\alpha] \longrightarrow A^*[[\beta_1, \beta_2]] \otimes E[\alpha_1, \alpha_2]$$

These are completely determined by:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto F\left(\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}, \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\right) \equiv \begin{pmatrix} F_1\left(\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}, \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\right) \\ F_2\left(\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}, \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\right)\end{pmatrix}$$

As in the Z/2 case, we require F to satisfy certain properties corresponding to the group structure of Z/p (c.f. 2.1.6).

Write γ_i for $\binom{\alpha_i}{\beta_i}$ $i = 1,2,3$.

(1) $F(0,\gamma) = F(\gamma,0) = \gamma$ -identity

(2) $F(\gamma_1, F(\gamma_2,\gamma_3)) = F(F(\gamma_1,\gamma_2),\gamma_3)$ -associativity

(3) $F(\gamma_1,\gamma_2) = F(\gamma_2,\gamma_1)$ -commutativity

(4) $[p]_F(\gamma) \underset{defn.}{\equiv} F(\underbrace{\gamma, F(\gamma, F(\gamma, \ldots}_{p})))) = 0$ -order p

We have an additional property corresponding to 3.3 :-

Let $h^*(-)$ be a representative Z/p-theory, so that we are given classes α_h, β_h, with

$$\begin{array}{ccc} h^*(BS^1) & \hookrightarrow & h^*(BZ/p) \\ \delta_h & \overset{i^*}{\longmapsto} & \beta_h \end{array}$$

$m: S^1 \times S^1 \to S^1$ induces $m: B(S^1 \times S^1) \simeq BS^1 \times BS^1 \to BS^1$ (since S^1 is abelian.)

This gives $m^*: h^*(BS^1) \to h^*(BS^1 \times BS^1)$

$$h^*[[\delta_h]] \to h^*[[\delta_h', \delta_h'']]$$

$$\delta_h \mapsto \tilde{F}(\delta_h', \delta_h'') \quad \text{say}$$

Thus $F_2\left(\binom{\alpha_1}{\beta_1}, \binom{\alpha_2}{\beta_2}\right) = i^*(\tilde{F}(\delta_h', \delta_h''))$ (since $i: Z/p \hookrightarrow S^1$ is a group homomorphism.)

so F_2 involves only β_1, β_2.

This motivates the property:

(5) $F_2\left(\binom{\alpha_1}{\beta_1}, \binom{\alpha_2}{\beta_2}\right)$ is independant of α_1 and α_2.

<u>Definition 3.4.1</u> An $F = \binom{F_1}{F_2}$ satisfying (1),(2),(3),(4),& (5) is called a <u>Z/p-formal group law</u>.

F_1 has the general form:

$$F_1\left(\binom{\alpha_1}{\beta_1}, \binom{\alpha_2}{\beta_2}\right) = \alpha_1 + \alpha_2 + \alpha_1 H_0(\beta_1,\beta_2) + \alpha_2 H_0'(\beta_1,\beta_2) + H_1(\beta_1,\beta_2) + \alpha_1\alpha_2 H_2(\beta_1,\beta_2)$$

(Note that (3) gives immediately that $H_0'(\beta_1,\beta_2) = H_0(\beta_2,\beta_1)$.)

F_2 has the general form:

$$F_2\left(\binom{\alpha_1}{\beta_1}, \binom{\alpha_2}{\beta_2}\right) = \beta_1 + \beta_2 + G_0(\beta_1,\beta_2).$$

Every representative Z/p-theory $h^*(-)$ (see 3.3.1) carries a Z/p-formal group law: $h^*(BZ/p) \longrightarrow h^*(BZ/p \times BZ/p)$ induced by

$$h^*(BZ/p) \cong h^*[[\beta_k]] \otimes E[\alpha_k]$$

the multiplication map: $Z/p \times Z/p \longrightarrow Z/p$.

<u>Definition 3.4.2</u> A <u>logarithm</u> l for F is a pair of series $l = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$:

$$l_1(\gamma) = l_1(\gamma) = \alpha + \sigma_1\beta + y_2\alpha\beta + \sigma_3\beta^2 + y_4\alpha\beta^2 + \ldots$$

$$l_2(\gamma) = l_2(\gamma) = \beta + x_2\beta^2 + x_4\beta^3 + x_6\beta^4 + \ldots$$

such that $l_i\begin{pmatrix} F_1(\gamma_1, \gamma_2) \\ F_2(\gamma_1, \gamma_2) \end{pmatrix} = l_i(\gamma_1) + l_i(\gamma_2) \quad (i = 1, 2)$

(l_2 involves only β because of (5).)

<u>Proposition 3.4.3</u> Let $h^*(-)$ be a representative Z/p-theory, with formal group law F_h, and suppose the ring epimorphism $\mu : h^*(-) \twoheadrightarrow H^*(-;Z/p)$ has a stable multiplicative natural splitting $\nu : H^*(-;Z/p) \longrightarrow h^*(-)$. If $\nu : H^*(BZ/p;Z/p) \longrightarrow h^*(BZ/p)$ is given by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto f\begin{pmatrix} \alpha_h \\ \beta_h \end{pmatrix} = \begin{pmatrix} f_1\binom{\alpha_h}{\beta_h} \\ f_2\binom{\alpha_h}{\beta_h} \end{pmatrix}$ then f is a logarithm for F_h. (c.f. 2.2.2)

<u>Proof</u> Because $H^*(pt.;Z/p) \cong Z/p$ is all in dimension 0, the Z/p-formal group law on $H^*(BZ/p;Z/p)$ is given by:

$$\alpha \longmapsto \alpha' + \alpha'' \qquad \beta \longmapsto \beta' + \beta''$$

Then precisely the same argument as 2.2.2 gives:

$$f\left(F_h\left(\binom{\alpha'_h}{\beta'_h}, \binom{\alpha''_h}{\beta''_h}\right)\right) = f\binom{\alpha'_h}{\beta'_h} + f\binom{\alpha''_h}{\beta''_h}$$

so f is a logarithm for F_h. ∎

<u>Remark 3.4.4</u> As in 2.2.1 we can prove that any ν is determined by its value on the space BZ/p. (All that is needed is the splitting principle used by Rourke ((34) or 3.1.2)) For the universal theory $V^*(-)$, constructed in later chapters, we can show, as in 2.2.3, that each logarithm f of F_V determines a natural ring splitting ν of $\mu : V^*(-) \twoheadrightarrow H^*(-;Z/p)$. The canonical logarithm (see 3.5.1 below) gives a canonical splitting .

3.5 The Universal Z/p-Formal Group

We use elaborations of methods of Lazard (25) to prove that every Z/p-formal group law has a logarithm. (The proof is long and complicated). The structure of the universal law is an easy corollary.

<u>Theorem 3.5.1</u> Let F be a Z/p-formal group law over ground ring A^*, and suppose

$$F_1\left(\binom{\alpha_1}{\beta_1},\binom{\alpha_2}{\beta_2}\right)=\alpha_1+\alpha_2+\alpha_1 H_0(\beta_1,\beta_2)+\alpha_2 H_0(\beta_2,\beta_1)+H_1(\beta_1,\beta_2)+ + \alpha_1\alpha_2 H_2(\beta_1,\beta_2)$$

$$F_2\left(\binom{\alpha_1}{\beta_1},\binom{\alpha_2}{\beta_2}\right)=\beta_1+\beta_2+G_0(\beta_1,\beta_2)$$

Then (i) $H_2(\beta_1,\beta_2)=0$

(ii) F has a logarithm, i.e. a pair of series $l=\begin{pmatrix} l_1 \\ l_2\end{pmatrix}$

$$l_1\binom{\alpha}{\beta}=\alpha+\sigma_1\beta+y_2\alpha\beta+\sigma_3\beta^2+y_4\alpha\beta^2+\dots$$

$$l_2\binom{\alpha}{\beta}=\beta+x_2\beta^2+x_4\beta^3+x_6\beta^4+\dots \qquad (\text{with } x_i,\ y_i,\ \sigma_i \in A^{-i})$$

such that $l_i\begin{pmatrix} F_1(\gamma_1,\gamma_2) \\ F_2(\gamma_1,\gamma_2)\end{pmatrix}=l_i(\gamma_1)+l_i(\gamma_2) \quad (i=1,2)$

(γ denotes $\binom{\alpha}{\beta}$)

This logarithm is unique if we require that $\left\{\begin{matrix}\sigma_{2i+1}=0 \text{ for } i=p^j-1 \\ x_{2i}=0 \text{ for } i=p^j-1\end{matrix}\right\}\forall j$

<u>Corollary 3.5.2</u> The universal Z/p-formal group law F_A lies over the ground ring A_V^* where:

$$A_V=P[x_{2i};i\neq p^j-1]\otimes E[\sigma_{2i+1};i\neq p^j-1]\otimes P[y_{2i};\text{ all } i>0]$$

$$\left(\begin{matrix} x_{2i} & \text{has grading} & -2i \\ \sigma_{2i+1} & " \quad " & -(2i+1) \\ y_{2i} & " \quad " & =2i \end{matrix}\right)$$

$F_A=\begin{pmatrix} F_1 \\ F_2\end{pmatrix}$ is the (unique) solution of:

$l_i\begin{pmatrix} F_1(\gamma_1,\gamma_2) \\ F_2(\gamma_1,\gamma_2)\end{pmatrix}=l_i(\gamma_1)+l_i(\gamma_2) \quad (i=1,2)$ where l_i are as above.

To prove 3.5.1 we need various lemmas and corollaries. 3.5.3 is adapted from Lazard.(see Fröhlich (19) for similar results).

3.5.4 is Lazard's Theorem.

3.5.5,6,7 are proved using techniques based on Lazard's methods.

<u>Lemma 3.5.3</u> Let A be an algebra over Z/p, and $\Gamma(X,Y)$ a homogeneous polynomial of degree n over A, then:

$$\Gamma \text{ satisfies } (*) \begin{cases} \text{(i)} & \Gamma(X,0)=\Gamma(0,X)=0 \\ \text{(ii)} & \Gamma(X,Y)+\Gamma(X+Y,Z)=\Gamma(X,Y+Z)+\Gamma(Y,Z) \\ \text{(iii)} & \Gamma(X,Y)=\Gamma(Y,X) \\ \text{(iv)} & \Gamma(X,X)+\Gamma(X,2X)+\dots+\Gamma(X,(p-1)X)=0 \end{cases}$$

$\Longleftrightarrow$ Γ is of the form $a([X+Y]^n-X^n-Y^n)$ for some $a\in A$.

<u>Proof</u> (a) Functions of the form $a([X+Y]^n-X^n-Y^n)$ satisfy $(*)$:-

(i),(iii),(iv) are immediate.

$$\text{l.h.s. of (ii)} = a([X+Y]^n-X^n-Y^n)+a([X+Y+Z]^n-[X+Y]^n-Z^n)$$
$$= a([X+Y+Z]^n-X^n-Y^n-Z^n) = \text{r.h.s. of (ii)}$$

by symmetry.

(b) If Γ satisfies $(*)$ then it is of the form $a([X+Y]^n-X^n-Y^n)$:-

Suppose $\Gamma=\sum_{i=0}^{n} a_i X^i Y^{n-i}$

(i) gives $a_0=a_n=0$

(iii) gives $a_i=a_{n-i}$

Equating coefficients of $X^iY^jZ^{n-(i+j)}$ in (ii) we obtain: $\quad a_{i+j}\binom{i+j}{j} = a_i\binom{n-i}{j}$

These give sufficient relations, for:

Case 1: n not divisible by p

If k is not divisible by p either, we have (putting i=1, j=k-1):-

$$a_k = \frac{1}{k}\binom{n-1}{k-1}a_1 = \frac{1}{n}\binom{n}{k}a_1 \ .$$

If k is divisible by p, then n-k is not, so:-

$$a_k = a_{n-k} = \frac{1}{n-k}\binom{n-1}{n-k-1}a_1 = \frac{1}{n}\binom{n}{n-k}a_1 \ .$$

Thus $\sum_{i=1}^{n-1} a_i X^i Y^{n-i} = \sum_{i=1}^{n-1} \frac{1}{n}\binom{n}{n-i} a_1 X^i Y^{n-i} = \frac{a_1}{n}([X+Y]^n - X^n - Y^n)$ as required.

Case 2: n=p

Again $a_k = \frac{1}{k}\binom{n-1}{k-1}a_1 \quad (1 \leqslant k \leqslant n-1)$

Thus $\Gamma(X,Y) = a_1\left(\frac{1}{p}([X+Y]^p - X^p - Y^p)\right)$ (division by p having an obvious meaning here)

Now (iv) gives:

$$a_1\left(\frac{1}{p}(\underbrace{[2X]^p - X^p - X^p} + \underbrace{[3X]^p - [2X]^p - X^p} + \ldots + \underbrace{[pX]^p - [(p-1)X]^p - X^p})\right) = 0.$$

i.e. $a_1\left(\frac{1}{p}\cdot pX^p\right) = 0$ so $a_1 = 0$.

Thus $\Gamma(X,Y) \equiv 0 \equiv a([X+Y]^p - X^p - Y^p)$ as required.

Case 3: n=pq (q>1)

$$a_{k+p-1}\binom{k+p-1}{p-1} = a_k\binom{n-k}{p-1} \ .$$

k=1 gives $a_1 = 0$.

For any k not divisible by p we have: $a_k = \frac{1}{k}\binom{n-1}{k-1}a_1 = 0$.

Thus $\Gamma(X,Y) \equiv \Gamma'(X^p, Y^p)$ where Γ' is a homogeneous polynomial, which clearly satisfies (i),(ii) & (iii) (but not obviously (iv)

If q is not divisible by p we apply Case 1 to Γ' (only (i), (ii) & (iii) were needed in the proof of Case1), and obtain:-

$$\Gamma(X,Y) \equiv \Gamma'(X^p, Y^p) \equiv a([X^p + Y^p]^q - X^{pq} - Y^{pq})$$
$$\equiv a([X+Y]^{pq} - X^{pq} - Y^{pq}) \text{ as required.}$$

If q is divisible by p we apply the above process to Γ' to get Γ'', and repeat until we reach a q not divisible by p;

the only remaining case is if we eventually arrive at a $\Gamma^{(r)}$ of degree $q = p$. (i.e. when $n = p^{r+1}$). Then we have, by the Case 2 argument:

$$\Gamma(X,Y) = \Gamma^{(r)}(X^{p^r}, Y^{p^r}) = a_1\left(\frac{1}{p}([X^{p^r}+Y^{p^r}]^p - X^{p^{r+1}} - Y^{p^{r+1}})\right)$$
$$a_1\left(\frac{1}{p}([X+Y]^{p^{r+1}} - X^{p^{r+1}} - Y^{p^{r+1}})\right)$$

and as in Case 2, Γ satisfying (iv) gives $a_1 = 0$ (whether or not $\Gamma^{(r)}$ satisfies (iv).)

Thus $\Gamma(X,Y) \equiv 0 \equiv a([X+Y]^{p^{r+1}} - X^{p^{r+1}} - Y^{p^{r+1}})$ as required. ▮

Corollary 3.5.4 (Lazard's Theorem) Let A be an algebra over Z/p, and $F_2(X,Y)$, a power series over A satisfying:

(1) $F_2(0,X) = F_2(X,0) = X$ -identity

(2) $F_2(X,F_2(Y,Z)) = F_2(F_2(X,Y),Z)$ -associativity

(3) $F_2(X,Y) = F_2(Y,X)$ -commutativity

(4) $[p]_{F_2}(X) \underset{defn.}{=} \underbrace{F_2(X,F_2(X,F_2(X,\ldots)))}_{p} = 0$ -order p

Then there is a series $l_2(X) = X + x_2X^2 + x_4X^3 + \ldots \in A[[X]]$ such that $l_2(F_2(X,Y)) = l_2(X) + l_2(Y)$. l_2 is unique if we require that $x_{2i} = 0$ for $i = p^j - 1$.

Proof Suppose $l_2^{(n-1)}$ has been constructed inductively to give:

$l_2^{(n-1)}(F_2(X,Y)) =_n l_2^{(n-1)}(X) + l_2^{(n-1)}(Y)$ ($=_n$ denotes = modulo terms of degree $\geq n$)

i.e. $l_2^{(n-1)}\{F_2[(l_2^{(n-1)})^{-1}(X), (l_2^{(n-1)})^{-1}(Y)]\} =_n X+Y$ .....(†)

The l.h.s. of (†) satisfies (1),(2),(3),(4) because F_2 does and they are easily seen to be properties preserved by power series transformations. Write l.h.s. of (†) as $G(X,Y)$:

Then $G(X,Y) =_{n+1} X+Y+\Gamma(X,Y)$ (Γ homogeneous of degree n)

Γ satisfies (i) & (iii) of 3.5.3 because G satisfies (1) & (3)

Γ satisfies (ii) of 3.5.3 because $G(X,Y)$ satisfies (2):-

i.e. $G(X,G(Y,Z)) = G(G(X,Y),Z)$

so $X+(Y+Z+\Gamma(Y,Z))+\Gamma(X,Y+Z+\Gamma(Y,Z)) =_{n+1} (X+Y+\Gamma(X,Y))+Z+\Gamma(X+Y+\Gamma(X,Y),Z)$

so $\Gamma(Y,Z)+\Gamma(X,Y+Z) =_{n+1} \Gamma(X,Y)+\Gamma(X+Y,Z)$

Γ satisfies (iv) of 3.5.3 because $G(X,Y)$ satisfies (4):-

$$G(\underbrace{X,G(X,G(X,\ldots}_{p})))=0$$

Now $\quad G(X,X) =_{n+1} X+X+\Gamma(X,X)$

so $\quad G(X,G(X,X)) =_{n+1} (X+X+\Gamma(X,X))+X+\Gamma(X,2X+\Gamma(X,X))$

$$=_{n+1} 3X+\Gamma(X,X)+\Gamma(X,2X)$$

...

$$0=G(X,\underbrace{G(X,G(X,\ldots}_{p}))) =_{n+1} pX+\Gamma(X,X)+\Gamma(X,2X)+\ldots+\Gamma(X,(p-1)X)$$

so Γ satisfies (iv).

Thus by lemma 3.5.3 $\Gamma = a_n([X+Y]^n - X^n - Y^n)$ for some $a_n \in A$.

Now put $f(X) = X - a_n X^n$ and we have: (since $G(X,Y) =_{n+1} X+Y+\Gamma(X,Y)$):

$$f(G(f^{-1}(X), f^{-1}(Y))) =_{n+1} X+Y \quad .$$

So, putting $l_2^{(n)}(X) = f(l_2^{(n-1)}(X))$ we have $l_2^{(n)}(F_2((l_2^{(n)})^{-1}(X),(l_2^{(n)})^{-1}(Y)))= \, =_{n+1} X+Y$.

i.e. $\quad l_2^{(n)}(F_2(X,Y)) =_{n+1} l_2^{(n)}(X) + l_2^{(n)}(Y)$ completing the induction.

The existence of l_2 follows by the completeness of $A[[X]]$.

The a_n are given uniquely in the proof above (by the proof of 3.5.3) except when $n = p^j$, in which case (again by the proof of 3.5.3) $\Gamma \equiv 0$, and we may then put $x_{2i} = 0$ for $i = p^j - 1$ and obtain a unique logarithm. ▮

<u>Corollary 3.5.5</u> Let A be an algebra over Z/p, $J(X,Y)$ a power series in X,Y over A, $F_2(X,Y)$ satisfying (1),(2),(3) & (4) of 3.5.4, then:

$$\text{J satisfies } (**) \begin{cases} \text{(i)} & J(X,0) = 0 = J(0,X) \\ \text{(ii)} & J(X,Y)+J(F_2(X,Y),Z) = J(Y,Z)+J(X,F_2(Y,Z)) \\ \text{(iii)} & J(X,Y) = J(Y,X) \\ \text{(iv)} & J(X,X)+J(X,F_2(X,X))+\dots \\ & \quad + J(X,\underbrace{F_2(X,F_2(X,\dots X))}_{p-1}) = 0 \end{cases}$$

$\Longleftrightarrow$ J is of the form: $f(F_2(X,Y))-f(X)-f(Y)$ for some power series $f(X) = a_1X+\dots \in A[[X]]$. The series is unique (up to multiplication by elements of A), if we require the coefficient of X^{p^j} to be zero for each j.

<u>Proof</u> (a) Functions of this form satisfy (**):-

(i),(iii),(iv) are immediate.

$$\begin{aligned} \text{l.h.s. of (ii)} &= f(F_2(X,Y))-f(X)-f(Y)+f(F_2(F_2(X,Y),Z)) \\ &\quad -f(F_2(X,Y))-f(Z) \\ &= f(F_2(F_2(X,Y),Z))-f(X)-f(Y)-f(Z) \end{aligned}$$

= r.h.s. of (ii) by symmetry (F_2 is associative).

(b) If J satisfies (**) then it has the required form:- Suppose $f_{(n-1)}$ has been constructed inductively to give:

$J =_n J_{(n-1)} \underset{defn.}{\equiv} f_{(n-1)}(F_2(X,Y))-f_{(n-1)}(X)-f_{(n-1)}(Y)$

$J_{(n-1)}$ satisfies (**) (by (a)) and J does by hypoth--esis.

Let $J =_{n+1} J_{(n-1)} + \Gamma(X,Y)$ (Γ homogeneous of degree n)

Γ satisfies (i) & (iii) of (*) 3.5.3 because J & $J_{(n-1)}$ satisfy (i) & (iii) of (**) (immediate)

Γ satisfies (ii)& (iv) of (*) 3.5.3 because J & $J_{(n-1)}$ satisfy (ii)& (iv) of (**) (almost immediate--using $F_2(X,Y) = X+Y+\dots$

Thus $\Gamma(X,Y) = a_n([X+Y]^n - X^n - Y^n)$ for some a_n by lemma 3.5.3.

Hence $J - J_{(n-1)} =_{n+1} a_n((F_2(X,Y))^n - X^n - Y^n)$, completing the induction step.

The uniqueness of the canonical form of f (with the coefficient of X^{p^j} vanishing) follows in exactly the same way as the last part of Corollary 3.5.4. ∎

Lemma 3.5.6 Let A be an algebra over Z/p, $\Gamma(X,Y)$ a homogeneous polynomial of degree n over A, then:

$$\Gamma \text{ satisfies } (\dagger) \quad \begin{cases} \text{(i)} & \Gamma(X,0) = 0 \\ \text{(ii)} & \Gamma(X,Y) + \Gamma(X+Y,Z) = \Gamma(X,Y+Z) \\ \text{(iii)} & \Gamma(Y,X) + \Gamma(X+Y,Z) = \Gamma(Y,Z) + \Gamma(Y+Z,X) \end{cases}$$

$\Longleftrightarrow$ Γ is of the form $a(X^n - [X+Y]^n)$ for some $a \in A$.

Proof (c.f. 3.5.3)

(a) Functions of the form $a(X^n - [X+Y]^n)$ satisfy $(\dagger)$ (by direct substitution).

(b) If Γ satisfies $(\dagger)$ then it is of the form $a(X^n - [X+Y]^n)$:-

Suppose $\Gamma = \sum_{i=0}^{n} a_i X^i Y^{n-i}$.

The term in X^n in (ii) gives: $a_n + a_n = a_n$ $\quad\therefore\ a_n = 0$.

Equating coefficients of $X^i Y^j Z^{n-(i+j)}$ gives us :

From (ii) $\quad a_{i+j}\binom{i+j}{j} = a_i\binom{n-i}{j} \quad (i+j \neq n)$

From (iii) $\quad a_{i+j}\binom{i+j}{j} = a_{n-i}\binom{n-i}{j} \quad (i+j \neq n,\ i \neq 0)$

Case 1: n not divisible by p

Putting $j = k-1$, $i = 1$, we get $\quad a_k\binom{k}{1} = a_1\binom{n-1}{k-1}$

Hence for k not divisible by p: $\quad a_k = \frac{1}{n}\binom{n}{k} a_1$ (from (ii

If k is divisible by p, then n-k is not, and we have:

$$a_k = \frac{1}{n}\binom{n}{n-k} a_1 \text{ (from (iii))}$$

Putting $j=1$, $i=0$, gives: $a_1 = na_0$.

So $\sum_{i=0}^{n} a_i X^i Y^{n-i} = \frac{a_1}{n}(X^n - [X+Y]^n)$ as required.

<u>Case 2: $n = p$</u>

Again, $a_k \binom{k}{1} = a_1 \binom{n-1}{k-1}$ $(1 \leqslant k \leqslant n-1)$

So $a_k = \frac{1}{k}\binom{n-1}{k-1} a_1$.

$j=1$, $i=0$, gives $a_1 = pa_0 = 0$.

So $\Gamma(X,Y) = a_0 Y^p$ as required.

<u>Case 3: $n = pq$ $(q > 0)$</u>

$j=1$, $i=0$, gives $a_1 = na_0 = 0$

For any k not divisible by p we have:

$a_k = \frac{1}{k}\binom{n-1}{k-1} a_1 = 0$

So Γ is of the form $\Gamma(X,Y) = \Gamma'(X^p, Y^p)$, where Γ' is homogeneous of degree q. It is immediate that satisfies (i),(ii),(iii) and the result follows by induction, using Case 1 and Case 2. ∎

<u>Corollary 3.5.7</u> Let A be an algebra over Z/p, $K(X,Y)$ a power series in X,Y over A, and $F_2(X,Y)$ a power series satisfying (1),(2),(3),(4) of 3.5.4, then:

$$\text{K satisfies } (\dagger\dagger) \begin{cases} \text{(i)} \quad K(X,0) = 0 \\ \text{(ii)} \quad K(X,Y) + K(F_2(X,Y),Z)(1+K(X,Y)) = \\ \qquad\qquad = K(X,F_2(Y,Z)) \\ \text{(iii)} \quad K(Y,X) + (1+K(Y,X))K(F_2(X,Y),Z) = \\ \qquad\qquad = K(Y,Z) + (1+K(Y,Z))K(F_2(Y,Z),X) \end{cases}$$

$$\Longleftrightarrow \quad \text{K is of the form: } \frac{f(X)}{f(F_2(X,Y))} - 1 \equiv \frac{1 + y_2 X + y_4 X^2 + y_6 X^3 + \dots\dots}{1 + y_2 F_2(X,Y) + y_4 (F_2(X,Y))^2 + \dots} - 1$$

for some $y_{2i} \in A$. Such y_{2i} are unique.

<u>Proof</u> (a) Functions of the form $\dfrac{f(X)}{f(F(X,Y))} - 1$ satisfy (††):-

(i) is obvious.

(ii) $$\text{l.h.s.} = \frac{f(X)}{f(F_2(X,Y))} - 1 + \left[\frac{f(F_2(X,Y))}{f(F_2(F_2(X,Y),Z))} - 1\right] \cdot \frac{f(X)}{f(F_2(X,Y))}$$

$$= \frac{f(X)}{f(F_2(F_2(X,Y),Z)} - 1 = \text{r.h.s.} \ (F_2 \text{ is associative})$$

(iii) $$\text{l.h.s.} = \frac{f(Y)}{f(F_2(Y,X))} - 1 + \frac{f(Y)}{f(F_2(Y,X))}\left[\frac{f(F_2(X,Y))}{f(F_2(F_2(X,Y),Z))} - 1\right]$$

$$= \frac{f(Y)}{f(F_2(F_2(X,Y),Z))} - 1$$

$$= \frac{f(Y)}{f(F_2(Y,Z))} + \frac{f(Y)}{f(F_2(Y,Z))}\left[\frac{f(F_2(Y,Z))}{f(F_2(F_2(Y,Z),X))} - 1\right]$$

$= \text{r.h.s}$ (F_2 is commutative & associative).

(b) If K satisfies (††) then it has the required form:-

Suppose $f_{(n-1)}$ has been constructed inductively to give:

$$K =_n K_{(n-1)} \underset{\text{defn.}}{\equiv} \frac{f_{(n-1)}(X)}{f_{(n-1)}(F_2(X,Y))} - 1$$

$K_{(n-1)}$ satisfies (††) by (a) and K satisfies (††) by hypothesis.

Let $K =_{n+1} K_{(n-1)} + \Gamma(X,Y)$ (Γ homogeneous of degree n).

Then Γ satisfies (i),(ii), & (iii) of (†) 3.5.6 because K and K satisfy (i),(ii) & (iii) of (††).

$(F_2(X,Y) = X + Y + \ldots)$

Thus $\Gamma = a(X^n - [X+Y]^n)$

So $$K =_{n+1} \frac{f_{(n-1)}(X) + aX^n}{f_{(n-1)}(F_2(X,Y)) + a(X+Y)^n} - 1 =_{n+1} \frac{f_{(n-1)}(X) + aX^n}{f_{(n-1)}(F_2(X,Y)) + a(F_2(X,Y))^n} - 1$$

Putting $f_{(n)}(X) = f_{(n-1)}(X) + aX^n$ completes the induction step. By the construction a is unique, and thus by completeness, in the limit f(X) is unique.

Proof of Theorem 3.5.1

(i): The associativity rule ((2) of 3.4) for F gives (on the F_1 term):-

$$F_1\left(\begin{pmatrix} F_1\left(\binom{\alpha_1}{\beta_1},\binom{\alpha_2}{\beta_2}\right) \\ F_2(\beta_1,\beta_2) \end{pmatrix}, \binom{\alpha_3}{\beta_3}\right) = F_1\left(\binom{\alpha_1}{\beta_1}, \begin{pmatrix} F_1\left(\binom{\alpha_2}{\beta_2},\binom{\alpha_3}{\beta_3}\right) \\ F_2(\beta_2,\beta_3) \end{pmatrix}\right)$$

Written out at length this is:

$$\begin{aligned} & F_1\left(\tbinom{\alpha_1}{\beta_1},\tbinom{\alpha_2}{\beta_2}\right)+\alpha_3+F_1\left(\tbinom{\alpha_1}{\beta_1},\tbinom{\alpha_2}{\beta_2}\right).H_0(F_2(\beta_1,\beta_2),\beta_3) \\ & +\alpha_3.H_0(\beta_3,F_2(\beta_1,\beta_2))+H_1(F_2(\beta_1,\beta_2),\beta_3) \\ & +F_1\left(\tbinom{\alpha_1}{\beta_1},\tbinom{\alpha_2}{\beta_2}\right).\alpha_3.H_2(F_2(\beta_1,\beta_2),\beta_3)=\alpha_1+F_1\left(\tbinom{\alpha_2}{\beta_2},\tbinom{\alpha_3}{\beta_3}\right)+\alpha_1H_0(\beta_1,F_2(\beta_2,\beta_3)) \\ & \qquad +F_1\left(\tbinom{\alpha_2}{\beta_2},\tbinom{\alpha_3}{\beta_3}\right).H_0(F_2(\beta_2,\beta_3),\beta_1)+H_1(\beta_1,F_2(\beta_2,\beta_3)) \\ & \qquad +\alpha_1F_1\left(\tbinom{\alpha_2}{\beta_2},\tbinom{\alpha_3}{\beta_3}\right).H_2(\beta_1,F_2(\beta_2,\beta_3)) \quad \ldots\ldots\ldots A \end{aligned}$$

Picking out from A the term in $\alpha_1\alpha_2$:-

$$\begin{aligned} & H_2(\beta_1,\beta_2)+H_2(\beta_1,\beta_2).\left[H_0(F_2(\beta_1,\beta_2),\beta_3)\right]= \\ & \qquad =\left[1+H_0(\beta_2,\beta_3)\right].H_2(\beta_1,F_2(\beta_2,\beta_3)) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots A_{1,2} \end{aligned}$$

Now $\alpha_1\alpha_2=-\alpha_2\alpha_1$, so the commutativity rule ((3) of 3.4) for F gives: $H_2(\beta_1,\beta_2)=-H_2(\beta_2,\beta_1)$ and in particular $H_2(\beta_2,\beta_2)=0$.

Thus, putting $\beta_1=\beta_2$ in $A_{1,2}$, we obtain:

$$(1+H_0(\beta_2,\beta_3)).H_2(\beta_2,F_2(\beta_2,\beta_3))=0$$

Suppose the lowest term in H_2 has degree n, i.e. :

$$H_2(\beta_2,\beta_3)\equiv_n \sum_{\substack{r_i+s_i=n \\ s_i>r_i}} a_i(\beta_2^{r_i}\beta_3^{s_i}-\beta_2^{s_i}\beta_3^{r_i})$$

H_0 has degree >0 (since F satisfies (1) of 3.4) so:

$$H_2(\beta_2,F_2(\beta_2,\beta_3))\equiv_n 0$$

Hence: $H_2(\beta_2,\beta_2+\beta_3)\equiv_n 0$.

$$\therefore \quad \sum_i a_i(\beta_2^{r_i}(\beta_2+\beta_3)^{s_i}-\beta_2^{s_i}(\beta_2+\beta_3)^{r_i})=0$$

$$\therefore \quad \sum_i a_i\beta_2^{r_i}((\beta_2+\beta_3)^{s_i}-\beta_2^{s_i-r_i}(\beta_2+\beta_3)^{r_i})=0$$

Take the least r_i with $a_i\neq 0$, say r_j, and divide by $\beta_2^{r_j}$; then the term in $\beta_3^{s_j}$ gives: $a_j\beta_3^{s_j}=0$. i.e. $a_j=0$. Thus all $a_i=0$ so $H_2(\beta_1,\beta_2)\equiv 0$.

(ii): Finding a logarithm is equivalent to solving:

$$l_1\begin{pmatrix} F_1\left(\binom{\alpha_1}{\beta_1},\binom{\alpha_2}{\beta_2}\right) \\ F_2(\beta_1,\beta_2) \end{pmatrix} = l_1\binom{\alpha_1}{\beta_1} + l_1\binom{\alpha_2}{\beta_2} \quad \ldots\ldots\ldots\ldots (I)$$

$$l_2(F_2(\beta_1,\beta_2)) = l_2(\beta_1) + l_2(\beta_2) \quad \ldots\ldots\ldots\ldots (II)$$

(1),(2),(3) & (4) for F ensure F_2 satisfies the conditions of Corollary 3.5.4 (Lazard's Theorem) and hence there is a unique solution to (II) of the required form, for l_2.

Writing out (I) at length:

$$F_1\left(\binom{\alpha_1}{\beta_1},\binom{\alpha_2}{\beta_2}\right)\left[1+y_2F_2(\beta_1,\beta_2)+y_4F_2(\beta_1,\beta_2)^2+\ldots\right]+\sigma_1F_2(\beta_1,\beta_2)+\sigma_3(F_2(\beta_1,\beta_2))^2+\ldots =$$

$$=\alpha_1(1+y_2\beta_1+y_4\beta_1^2+\ldots)+\sigma_1\beta_1+\sigma_3\beta_1^2+\ldots+\alpha_2(1+y_2\beta_2+y_4\beta_2^2+\ldots)+\sigma_1\beta_2+\sigma_3\beta_2^2+\ldots \quad \ldots\ldots (I')$$

Solving (I′) is equivalent to solving two equations (one for the term independent of α_1 & α_2 and one for the term in α_1; that in α_2 is the same as for α_1)

As usual, put $F_1\left(\binom{\alpha_1}{\beta_1},\binom{\alpha_2}{\beta_2}\right)=\alpha_1+\alpha_2+\alpha_1H_0(\beta_1,\beta_2)+\alpha_2H_0(\beta_2,\beta_1)+H_1(\beta_1,\beta_2)$ in (I′)

$(H_2 \equiv 0)$

Then the coefficients of α_1 in (I′) give us to solve:

$$[1+H_0(\beta_1,\beta_2)][1+y_2F_2(\beta_1,\beta_2)+\ldots] = 1+y_2\beta_1+\ldots \quad \ldots\ldots\ldots\ldots\ldots (III)$$

The coefficients independent of α_1 & α_2 in (I′) give us to solve:

$$H_1(\beta_1,\beta_2)\left[1+y_2F_2(\beta_1,\beta_2)+y_4(F_2(\beta_1,\beta_2))^2+\ldots\right]+\sigma_1F_2(\beta_1,\beta_2)+\sigma_3(F_2(\beta_1,\beta_2))^2+\ldots$$
$$=\sigma_1\beta_1+\sigma_3\beta_1^2+\ldots+\sigma_1\beta_2+\sigma_3\beta_2^2+\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (IV)$$

The proof now proceeds by showing that the formal group law rules for F provide sufficient conditions to give a unique solution of (III) & (IV) for y_{2i}, σ_{2i+1} of the required form. We assemble the neceesary data:

The term of A independent of α_1 & α_2 gives:-

$$H_1(\beta_1,\beta_2)+H_1(F_2(\beta_1,\beta_2),\beta_3)+H_1(\beta_1,\beta_2).H_0(F_2(\beta_1,\beta_2),\beta_3)=$$
$$= H_1(\beta_2,\beta_3)+H_1(\beta_1,F_2(\beta_2,\beta_3))+H_1(\beta_2,\beta_3).H_0(F_2(\beta_2,\beta_3),\beta_1) \quad \ldots \textbf{(A1)}$$

The term of A in α_1 gives:

$$H_0(\beta_1,\beta_2)+H_0(F_2(\beta_1,\beta_2),\beta_3)(1+H_0(\beta_1,\beta_2)) = H_0(\beta_1,F_2(\beta_2,\beta_3)) \quad \ldots\ldots\ldots\ldots\ldots(A2)$$

(The term of A in α_3 is the same, by symmetry.)

The term of A in α_2 gives:

$$H_0(\beta_2,\beta_1)+(1+H_0(\beta_2,\beta_1)).H_0(F_2(\beta_1,\beta_2),\beta_3) = H_0(\beta_2,\beta_3)+(1+H_0(\beta_2,\beta_3))H_0(F_2(\beta_2,\beta_3),\beta_1) \ldots(A3$$

Also $F_1((\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}),0) = \alpha$ (by condition (1) of 3.4)

So $H_0(\beta,0) = 0 \quad \ldots\ldots\ldots\ldots(B1)$

& $H_1(\beta,0) = 0 \quad \ldots\ldots\ldots\ldots(B2)$

(A2).(A3),(B1) give that $H_0(X,Y)$ satisfies the conditions (i), (ii), & (iii) of Corollary 3.5.7. Thus $H_0(\beta_1,\beta_2)$ is uniquely expressible as:

$$\frac{1+y_2\beta_1+y_4\beta_1^2+\ \ldots}{1+y_2F_2(\beta_1,\beta_2)+y_4(F_2(\beta_1,\beta_2))^2+\ldots}-1$$

i.e. we have a unique solution to (III).

It remains to solve (IV) for σ_{2i+1}.

Writing $\sigma_1\beta+\sigma_3\beta^3+\ldots$ as $h(\beta)$, (IV) is:

$$H_1(\beta_1,\beta_2)\left[1+y_2F_2(\beta_1,\beta_2)+y_4(F_2(\beta_1,\beta_2))^2+\ldots\right] = h(\beta_1)+h(\beta_2)-h(F_2(\beta_1,\beta_2)) \quad \ldots\ldots(IV')$$

Call the l.h.s. of (IV$'$) $J(\beta_1,\beta_2)$.

Now 3.3.5 gives us a unique solution of (IV$'$) for h if and only if $J(\beta_1,\beta_2)$ satisfies (**) of 3.5.5, so this is all that it remains to check to complete the proof of 3.5.1.

(B2) gives that $J(\beta_1,\beta_2)$ satisfies (i) of (**).

$F_1((\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}),(\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix})) = F_1((\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}),(\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}))$ (by condition (3) of 3.4)

so that $J(\beta_1,\beta_2)$ satisfies (iii) of (**).

<u>To show $J(\beta_1,\beta_2)$ satisfies (**)(ii):</u>

Substituting $H_0(\beta_1,\beta_2) = \dfrac{1+y_2\beta_1+\ldots}{1+y_2F_2(\beta_1,\beta_2)+\ldots}-1$ in (A1) we have:

$$\text{l.h.s} = H_1(\beta_1,\beta_2)\left[1+\frac{1+y_2F_2(\beta_1,\beta_2)+\ldots}{1+y_2(F_2(F_2(\beta_1,\beta_2),\beta_3))+\ldots}-1\right]+H_1(F_2(\beta_1,\beta_2),\beta_3)$$

$$= \frac{1}{1+y_2(F_2(F_2(\beta_1,\beta_2),\beta_3))+\ldots}.\left[J(\beta_1,\beta_2)+J(F_2(\beta_1,\beta_2),\beta_3)\right]$$

Similarly r.h.s $= \dfrac{1}{1+y_2(F_2(\beta_1,F_2(\beta_2,\beta_3)))+\dots}\left[J(\beta_2,\beta_3)+J(\beta_1,F_2(\beta_2,\beta_3))\right]$

As F_2 is associative, the denominators are equal, and putting l.h.s of (A1) = r.h.s of (A1) gives us that $J(\beta_1,\beta_2)$ satisfies (**)(ii).

<u>To show that $J(\beta_1,\beta_2)$ satisfies (**)(iv)</u>

$$\begin{pmatrix}[n]_F(\gamma)_1\\ [n]_F(\gamma)_2\end{pmatrix} \underset{defn.}{\equiv} [n]_F(\gamma) \underset{defn.}{\equiv} F(\underbrace{\gamma,F(\gamma,F(\gamma,\dots}_{n}))) \qquad (\gamma=\binom{\alpha}{\beta})$$

(so $[n]_F(\gamma)_2 = [n]_{F_2}(\beta)$)

$[n]_F(\gamma) = F(\gamma,[n-1]_F(\gamma))$ gives:

$$[n]_F(\gamma)_1 = \alpha + [n-1]_F(\gamma)_1 + \alpha H_0(\beta,[n-1]_{F_2}(\beta)) + [n-1]_F(\gamma)_1 . H_0([n-1]_{F_2}(\beta),\beta) + H_1(\beta,[n-1]_{F_2}(\beta))$$

Let $G_n(\beta)$ be the term of $[n]_F(\gamma)_1$ not involving α.

Then: $G_n(\beta) = G_{n-1}(\beta) + G_{n-1}(\beta).H_0([n-1]_{F_2}(\beta),\beta) + H_1(\beta,[n-1]_{F_2}(\beta))$.

Substituting $H_0(\beta_1,\beta_2) = \dfrac{1+y_2\beta_1+\dots}{1+y_2F_2(\beta_1,\beta_2)+\dots} - 1$ we get:

$$G_n(\beta) = G_{n-1}(\beta)\left(\frac{1+y_2([n-1]_{F_2}(\beta))+\dots}{1+y_2([n]_{F_2}(\beta))+\dots}\right) + H_1(\beta,[n-1]_{F_2}(\beta))$$

Thus
$$G_n(\beta) = \frac{1}{1+y_2([n]_{F_2}(\beta))+\dots}\Big[(1+y_2([n]_{F_2}(\beta))+\dots).H_1(\beta,[n-1]_{F_2}(\beta)) + (1+y_2([n-1]_{F_2}(\beta))+\dots).H_1(\beta,[n-2]_{F_2}(\beta)) + \dots + (1+y_2([2]_{F_2}(\beta))+\dots).H_1(\beta,\beta)\Big]$$

So
$$G_p(\beta) = \frac{1}{1+y_2([p]_{F_2}(\beta))+\dots}\Big[J(\beta,[p-1]_{F_2}(\beta)) + J(\beta,[p-2]_{F_2}(\beta)) + \dots + J(\beta,\beta)\Big]$$

But (4) of 3.4 gives: $[p]_F(\gamma) = 0$, so $G_p(\beta) = 0$.

Thus J satisfies (iv) of (**) of 3.5.5 , completing the proof of 3.5.1. ∎

Proof of Corollary 3.5.2

It is sufficient to check the equations

$l_i\begin{pmatrix} F_1(\gamma_1,\gamma_2) \\ F_2(\gamma_1,\gamma_2)\end{pmatrix} = l_i(\gamma_1) + l_i(\gamma_2)$ $(i = 1,2)$ do indeed define a unique Z/p-formal group law $F_A = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$.

(For then, given any Z/p- formal group (A,F) we have a canonical ring homomorphism $\phi: A_V \to A$ under which the coefficients $x_{2i}, y_{2i}, \sigma_{2i+1}$ of $\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ map to the corresponding coefficients of the canonical logarithm for F, and this map ϕ is the unique ring map sending F_A to F ; x_{2i} & y_{2i} poly. and σ_{2i+1} ext. give the largest ring allowed by the commutativity conditions)

We need to show first that $\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ has a unique inverse $l^{-1} = \begin{pmatrix} (l^{-1})_1 \\ (l^{-1})_2 \end{pmatrix}$ such that $l^{-1}(l) = \text{id}$. i.e. such that

$$(l^{-1})_1\begin{pmatrix} l_1\binom{\alpha}{\beta} \\ l_2\binom{\alpha}{\beta}\end{pmatrix} = \alpha \quad \ldots\ldots \quad (i)$$

$$(l^{-1})_2\begin{pmatrix} l_1\binom{\alpha}{\beta} \\ l_2\binom{\alpha}{\beta}\end{pmatrix} = \beta \quad \ldots\ldots \quad (ii)$$

Consider (ii): $\left(l_1\binom{\alpha}{\beta}\right)^2 = 0$ and $l_1\binom{\alpha}{\beta} = \alpha + \ldots$ so $(l^{-1})_2\binom{\alpha}{\beta}$ is independent of α. Thus $(l^{-1})_2$ is uniquely defined and is just l_2^{-1}. (since $l_2(X) = X + \ldots$, l_2 has a unique inverse series l_2^{-1}).

Consider (i): let $g\binom{\alpha}{\beta} = \alpha + a_1\beta + b_2\alpha\beta + a_3\beta^2 + b_4\alpha\beta^2 + \ldots$

$$= \alpha(1 + b_2\beta + b_4\beta^2 + \ldots) + a_1\beta + a_3\beta^2 + \ldots$$

Then $g\begin{pmatrix} l_1\binom{\alpha}{\beta} \\ l_2\binom{\alpha}{\beta}\end{pmatrix} = (\alpha + \sigma_1\beta + y_2\alpha\beta + \sigma_3\beta^2 + y_4\alpha\beta^2 + \ldots)(1 + b_2 l_2(\beta) + b_4(l_2(\beta))^2 + \ldots)$

$$+ a_1 l_2(\beta) + a_3(l_2(\beta))^2 + \ldots$$

Thus $g\begin{pmatrix} l_1\binom{\alpha}{\beta} \\ l_2\binom{\alpha}{\beta}\end{pmatrix} = \alpha \iff (1 + y_2\beta + y_4\beta^2 + \ldots)(1 + b_2 l_2(\beta) + b_4(l_2(\beta))^2 + \ldots) = 1$

$$\ldots\ldots\ldots \quad (iii)$$

$$\& \quad (\sigma_1\beta + \sigma_3\beta^2 + \ldots)(1 + b_2 l_2(\beta) + \ldots) + a_1 l_2(\beta) + \ldots = 0$$

$$\ldots\ldots\ldots \quad (iv)$$

Since $l_2(\beta)=\beta+\ldots$ we can use (iii) to define b_{2i} inductively

(iv) to define a_{2i+1} inductively

These are unique, and so we have a unique inverse l^{-1} to l,

i.e. $F_A=\begin{pmatrix}F_1\\F_2\end{pmatrix}$ is uniquely defined by $F_A\left(\begin{pmatrix}\alpha_1\\\beta_1\end{pmatrix},\begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix}\right)=l^{-1}\left(l\begin{pmatrix}\alpha_1\\\beta_1\end{pmatrix}+l\begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix}\right)$

It only remains to check that F_A satisfies (1),(2),(3), (4) & (5) of 3.4 and so is a Z/p-formal group law. This is straightforward, e.g. for (2):-

$$\begin{aligned}F_A(F_A(\gamma_1,\gamma_2),\gamma_3)&=l^{-1}\left(l\left(l^{-1}\left(l\begin{pmatrix}\alpha_1\\\beta_2\end{pmatrix}+l\begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix}\right)\right)+l\begin{pmatrix}\alpha_3\\\beta_3\end{pmatrix}\right)\\&=l^{-1}\left(l\begin{pmatrix}\alpha_1\\\beta_1\end{pmatrix}+l\begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix}+l\begin{pmatrix}\alpha_3\\\beta_3\end{pmatrix}\right)\\&=F_A(\gamma_1,F_A(\gamma_2,\gamma_3))\end{aligned}$$

The other conditions are just as easy. ∎

Remark 3.5.8

Consider the cohomology theory $H^*(-;Z/p)\otimes A_V=h^*(-)$. If l is the canonical logarithm of F_A on A_V, and

$$l^{-1}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}\alpha+\beta a_1+\alpha\beta b_2+\beta^2 a_3+\ldots\\\beta+\beta^2 c_2+\beta^3 c_4+\ldots\end{pmatrix}$$

put $\begin{pmatrix}\alpha_h\\\beta_h\end{pmatrix}=\begin{pmatrix}\alpha\otimes 1+\beta\otimes a_1+\alpha\beta\otimes b_2+\ldots\\\beta\otimes 1+\beta^2\otimes c_2+\beta^3\otimes c_4+\ldots\end{pmatrix}\in\begin{pmatrix}h^1(BZ/p)\\h^2(BZ/p)\end{pmatrix}$

This then gives an "algebraic" realisation of the universal Z/p-formal group law as a cohomology theory, since under $m: BZ/p\times BZ/p\to BZ/p$ we have:

$$m^*:\begin{pmatrix}\alpha\\\beta\end{pmatrix}\longmapsto\begin{pmatrix}\alpha'+\alpha''\\\beta'+\beta''\end{pmatrix}=l\begin{pmatrix}\alpha'_h\\\beta'_h\end{pmatrix}+l\begin{pmatrix}\alpha''_h\\\beta''_h\end{pmatrix}$$

$$\text{so}\quad m^*:\begin{pmatrix}\alpha_h\\\beta_h\end{pmatrix}=l^{-1}\begin{pmatrix}\alpha\\\beta\end{pmatrix}\longmapsto l^{-1}\left\{l\begin{pmatrix}\alpha'_h\\\beta'_h\end{pmatrix}+l\begin{pmatrix}\alpha''_h\\\beta''_h\end{pmatrix}\right\}\underset{\text{defn.}}{=}F_A\left\{\begin{pmatrix}\alpha'_h\\\beta'_h\end{pmatrix},\begin{pmatrix}\alpha''_h\\\beta''_h\end{pmatrix}\right\}$$

However, what we would like is a geometric theory $V^*(-)$ where the α_V & β_V are naturally occuring "Euler classes" (as with e_N in the Z/2-case). This will be developed in the following sections.