A holomorphic correspondence at the boundary of the Klein combination locus

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Abstract

We investigate an explicit holomorphic correspondence on the Riemann sphere with striking dynamical behaviour: the limit set is a fractal resembling the one-skeleton of a tetrahedron and on each component of the complement of this set the correspondence behaves like a Fuchsian group.

1 Introduction

A one (complex) parameter family of holomorphic correspondences $\mathcal{F}_a$ containing matings between the modular group and quadratic polynomials was discovered by the first author and Christopher Penrose nearly twenty years ago [4], and further investigated in [3] and [2]. Two naturally defined subsets of interest in the parameter space are:

1. The *connectivity locus* $\mathcal{M}$.

2. The *Klein combination locus* $\mathcal{K} \supset \mathcal{M}$.

Both loci will be given precise mathematical definitions below, in Section 2. But we mention now that $\mathcal{M}$ is conjectured to be the set of values of a such that $\mathcal{F}_a$ is a mating between $PSL_2(\mathbb{Z})$ and $q_c : z \to z^2 + c$ for some $c \in M$, the Mandelbrot set, and that $\mathcal{K}$ is the interior of the set of values of a for which the action of $\mathcal{F}_a$ is 'faithful and discrete', in a sense not yet formalised. The set $\mathcal{M}$ is conjectured to be homeomorphic to the Mandelbrot set and the set $\mathcal{K}$ is conjectured to be homeomorphic to a disc.

The correspondences $\mathcal{F}_a$ for $a \in \mathcal{K} \setminus \mathcal{M}$ may be thought as 'matings between $PSL_2(\mathbb{Z})$ and maps q_c for which the Julia set is a Cantor set'. A classification of these correspondences will be presented in [5]. Any two such $\mathcal{F}_a$ are quasiconformally conjugate, provided each has no *critical relations*. The exceptional

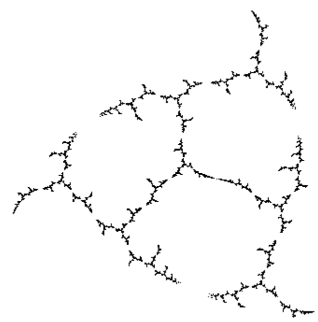

Figure 1: The limit set of the correspondence $\mathcal{F}_{a_{1/3}}$. The 'gaps' are an illusion arising from the presence of parabolic points.

values of a, for which $\mathcal{F}_a$ has some critical relation, form a countable set of isolated points in $\mathcal{K} \setminus \mathcal{M}$. Our interest in the current article is in the behaviour of $\mathcal{F}_a$ as a approaches the outer boundary of $\mathcal{K}$ and what happens when a reaches that boundary. A good analogy is the behaviour of a one (complex) parameter family of Kleinian groups, representations $G \to PSL_2(\mathbb{C})$ of a finitely generated abstract group G indexed by a parameter a, as a approaches the boundary of the slice in which the representations are discrete and faithful. In this paper we shall focus on one particular boundary point of $\mathcal{K}$, which we call the *Penrose point*, and investigate the dynamical behaviour (Figure 1) of the correspondence which has this as its parameter value, in particular showing that the complement of the limit set has four components and that for each component the subcorrespondence which stabilises it is conjugate to a Fuchsian group. More general results and conjectures concerning correspondences in the family $\mathcal{F}_a$, and the structure of $\mathcal{K}$ and its boundary, will be presented elsewhere.

2 The family of matings, the connectivity locus $\mathcal{M}$, and the Klein combination locus $\mathcal{K}$

In this section we briefly introduce the family of holomorphic correspondences $\mathcal{F}_a$ in which we are interested. For more details the reader should consult the original paper [4].

The first ingredient in the definition of $\mathcal{F}_a$ is the notion of a covering correspondence. For a given rational map $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the associated *covering correspondence* Cov^q is defined by the relation

$$(z, w) \in Cov^q \Leftrightarrow q(z) = q(w).$$

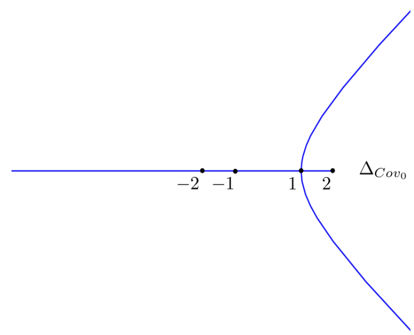

Figure 2: Fundamental domains for Cov_0: the curves are the preimage of $[-\infty, 2]$ under $Q(z) = z^3 - 3z$.

For each $z \in \hat{\mathbb{C}}$, by letting $Cov^q(z) = \{w : (z,w) \in Cov^q\}$ we can consider the correspondence to be a multifunction. Doing so allows us to consider correspondences from a dynamical perspective. Taking this point of view it is often convenient to consider the associated *deleted covering correspondence* Cov_0^q defined by

$$(z,w) \in Cov_0^q \Leftrightarrow \frac{q(z) - q(w)}{z - w} = 0.$$

Noting that all rational maps are branched covering maps of the sphere to itself we make use of branch cuts to understand how Cov^q acts as a transformation of the Riemann sphere. For the purposes of the current article we make the following definition of a *fundamental domain* for this action.

Definition 1 *A fundamental domain for the action of Cov^q is any component of $\hat{\mathbb{C}} \setminus q^{-1}(\ell)$, where ℓ is any piecewise smooth simple arc which starts and ends at two of the critical values of q and passes through all the others.*

We note that if Δ is a fundamental domain for Cov^q then Δ is mapped bijectively onto each of the other components of $\hat{\mathbb{C}} \setminus q^{-1}(\ell)$ by Cov_0^q.

Up to conjugacy there are only two possibilities for covering correspondences of cubic *polynomials*. This is because in addition to the critical point ∞, the polynomial either has a double critical point (which we may take to be at 0) or two simple critical points. We restrict attention to the second case and to the particular cubic polynomial $Q(z) = z^3 - 3z$. The critical points of Q are $-1, 1$ and ∞, and the corresponding critical values are $2, -2$ and ∞. The preimages of 2 are -1 and 2, the preimages of -2 are 1 and -2, and the only preimage of ∞ is ∞. Consider the preimage under Q of the real interval $[-\infty, 2]$ (see Figure 2). One possible choice of fundamental domain Δ_{Cov} is the right-hand component in Figure 2, with boundary the curves running from 1 to ∞, at asymptotic angles $\pm\pi/3$, together with the cut from 1 to 2. The correspondence Cov_0^Q

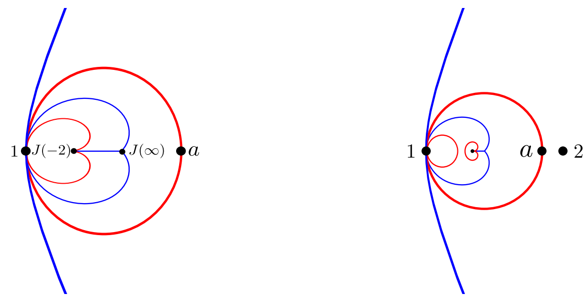

Figure 3: D_J and $D_1 = \mathcal{F}_a(D_J)$, pictured on the left for $a \in (2, 7]$ and on the right for $a \in (1, 2)$.

maps the cut $[1, 2]$ one-to-two onto the interval $[-2, 1]$, sending 1 to -2 and to 1, and sending 2 to the single point, -1. The point -2 also has a unique image under Cov_0^Q, namely 1, but all other points of $\hat{\mathbb{C}}$ (except ∞) have two distinct images.

The $(2:2)$ correspondences $\mathcal{F}_a$ with which we shall be concerned are defined for all $a \in \hat{\mathbb{C}}$ with $a \neq 1$. We set $\mathcal{F}_a$ to be the composition $J \circ Cov_0^Q$ where $J = J_a$ is the (unique) involution of $\hat{\mathbb{C}}$ which has fixed points 1 and a. Viewing J as Cov_0^q where q is the projection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}/J$, a *fundamental domain* Δ_J for J (in the sense of Definition 1) is a component of the complement of any piecewise smooth simple closed curve which passes through 1 and a and is invariant under J (for example, any round circle passing through 1 and a).

For certain values of the parameter a, there will exist fundamental domains Δ_J and Δ_{Cov} which satisfy the following *Klein combination condition*:

$$(K) \qquad \Delta_{Cov} \cup \Delta_J = \hat{\mathbb{C}} \setminus \{1\}.$$

Recall that we have defined our fundamental domains to be open sets, so the point 1, being on the boundary of both Δ_{Cov} and Δ_J, cannot lie in their union. Note also that condition (K) implies that $\Delta_{Cov} \cap \Delta_J \neq \emptyset$ since $\hat{\mathbb{C}} \setminus \{1\}$ is connected.

We can satisfy condition (K) by, for example, taking a in the half-open real interval $(1, 7]$. Choose as Δ_{Cov} the right-hand component in Figure 2. For any value of $a \in (1, 7]$ the circle C which passes through 1 and a and has centre on the real axis meets the boundary of Δ_{Cov} at the single point 1. If we take as Δ_J the component of $\hat{\mathbb{C}} \setminus C$ containing ∞ then the pair Δ_{Cov}, Δ_J satisfy the condition (K).

When condition (K) is satisfied, the closed disc D_J complementary to Δ_J is mapped 1 to 2 into itself by the multifunction $\mathcal{F}_a$. The image $D_1 = \mathcal{F}_a(D_J) \subset D_J$ of D_J will be either one topological disc or two, depending on whether the critical value 2 of the map $\mathcal{F}_a^{-1}$ lies in D_J or does not (see Figure 3). In either case, writing Δ for $\Delta_{Cov} \cap \Delta_J$, we observe that $D_J \setminus D_1$ consists of the union (modulo boundaries) of three 'tiles' which are homeomorphic copies of Δ, namely $J(\Delta)$ and the two images $J \circ Cov_0(\Delta)$ of Δ. These three tiles together make up $J \circ Cov(\Delta)$. Writing Λ_+ for $\bigcap_{n=0}^{\infty} \mathcal{F}_a^n(D_J)$, the 'ping-pong principle' underlying the Klein combination theorem for covering correspondences, [1], ensures that $D_J \setminus \Lambda_+$ is tiled by images of Δ under iterates of $J \circ Cov_0 (= \mathcal{F}_a)$ applied to these three tiles. Similarly, writing Λ_- for $J(\Lambda_+)$, the set $J(D_J) \setminus \Lambda_-$ is tiled by images of $Cov(\Delta)$ under iterates of $Cov_0 \circ J (= \mathcal{F}_a^{-1})$. To state this more precisely, setting Λ to be the union $\Lambda = \Lambda_+ \cup \Lambda_-$ (which is closed and invariant under $\mathcal{F}_a$ and $\mathcal{F}_a^{-1}$), and setting Ω to be its complement in $\hat{\mathbb{C}}$, the Klein combination theorem for covering correspondences tells us that:

1. The action of $\mathcal{F}_a$ on Ω is properly discontinuous and is the free product of the actions of Cov and $\{Id, J\}$, in an appropriate sense (see [1]);

2. $\Delta = \Delta_{Cov} \cap \Delta_J$ is a fundamental domain for this action, in the sense that $\Omega = \bigcup_{n \in \mathbb{Z}} \mathcal{F}_a^n(\Delta')$ where $\Delta' = \Delta \cup (\partial\Delta \setminus \{1\})$ and the 'tiles' (the images of Δ' under single-valued restrictions of the $\mathcal{F}_a^n$) meet only at their boundaries.

Remark Theorem 2 of [1] states this result for the simpler case that $\Delta_{Cov} \cup \Delta_J = \hat{\mathbb{C}}$, rather than $\Delta_{Cov} \cup \Delta_J = \hat{\mathbb{C}} \setminus \{1\}$, and is phrased in terms of transversals rather than fundamental domains, but the proof in [1] can be adapted to prove the two statements above.

Definition 2 *The Klein combination locus for the family $\mathcal{F}_a$ is the set $\mathcal{K}$ of values of $a \in \hat{\mathbb{C}}$ such that there exist fundamental domains Δ_{Cov}, Δ_J for Cov^Q and J satisfying condition (K).*

When condition (K) is satisfied, and $2 \in D_J$, the single-valued map $\mathcal{F}_a^{-1} : D_1 \to D_J$ is a *pinched quadratic-like map* with pinch point 1, critical point $J(-1)$ and critical value 2, and for certain values of a (see [4, 3, 2]) $\mathcal{F}_a$ is a mating in the sense of Definition 3 below.

Represent the action of the modular group $PSL_2(\mathbb{Z})$ on the complex upper half-plane by the *modular correspondence*,

$$(z, w) \in F_{Mod} \Leftrightarrow ((\tau_1(z) - w)(\tau_2(z) - w) = 0.$$

where

$$\tau_1(z) = z + 1 \qquad \text{and} \qquad \tau_2(z) = \frac{z}{z+1}.$$

Definition 3 *Let $q_c : z \to z^2 + c$ be a quadratic polynomial with connected filled Julia set K_c. We say that a $(2:2)$ holomorphic correspondence F is a mating between q_c and the modular group $PSL_2(\mathbb{Z})$ if:*

1. *There exist an open subset Ω of $\hat{\mathbb{C}}$ invariant under the action of F, and a conformal homeomorphism h from Ω to the upper-half plane conjugating the action of F to F_{Mod}.*

2. *The complement of Ω in $\hat{\mathbb{C}}$ is the union of two sets Λ_+ and Λ_-, with $\Lambda_+ \cap \Lambda_-$ consisting of a single point, and there exist homeomorphisms $h_+ : \Lambda_+ \to K_c$ and $h_- : \Lambda_- \to K_c$, which are conformal on interiors and respectively conjugate the action of F^{-1} restricted to Λ_+, and that of F restricted to Λ_-, to the action of q_c on K_c.*

For $a \in \mathcal{K}$, the set $\Lambda_+ = \bigcap_{n=1}^{\infty} \mathcal{F}_a^n(D_J)$ is connected if and only if the critical value 2 of the pinched quadratic-like map $(\mathcal{F}_a)^{-1} : D_1 \to D_J$ lies in $(\mathcal{F}_a)^n(D_J)$ for all $n > 0$. (For a proof see [4], or the Remark following the proof of Proposition 1 in Section 3 below.)

Definition 4 *The connectivity locus for the family $\mathcal{F}_a$ is the subset $\mathcal{M} \subset \mathcal{K}$ of values of a for which there is a pair of fundamental domains Δ_{Cov}, Δ_J satisfying the condition (K), and with the additional property that $2 \in \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$ (where $D_J = \hat{\mathbb{C}} \setminus \Delta_J$).*

A computer plot of the connectivity locus was presented in the original paper on this family of correspondences [4]. It bears a striking resemblance to the Mandelbrot set M for quadratic maps. It is conjectured that the family $\mathcal{F}_a$ contains a mating of q_c with $PSL_2(\mathbb{Z})$ for every value of $c \in M$ and that $\mathcal{M}$ is homeomorphic to M. The first conjecture was proved for a large subset of values of $c \in M$ in [2]. Our interest in the current article is in $\mathcal{K} \setminus \mathcal{M}$, and in particular a specific point on the boundary of $\mathcal{K}$. The way that we have defined $\mathcal{K}$ ensures that $\mathcal{K} \setminus \mathcal{M}$ is an open subset of the parameter space, and thus that the boundary points are outside $\mathcal{K}$.

Remark There is experimental evidence which suggests that for all a outside the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$, the periodic points of $\mathcal{F}_a$ are dense in $\hat{\mathbb{C}}$. It is tempting to hope that by analogy with the study of deformations of Kleinian groups there might be some definition of 'faithful and discrete' for a correspondence action which would allow us to characterise $\overline{\mathcal{K}}$ as the set of all values of $a \in \hat{\mathbb{C}}$ for which $\mathcal{F}_a$ satisfies this yet to be formulated property.

3 Dynamics of the correspondence $\mathcal{F}_a$ when a lies in $\mathcal{K} \setminus \mathcal{M}$

When $a \in \mathcal{K} \setminus \mathcal{M}$ we still have the 'Klein combination' set-up for Δ_{Cov} and Δ_J, and we still have a partition of the Riemann sphere into invariant regions Ω and $\Lambda = \Lambda_+ \cup \Lambda_-$, but the filled Julia set $\Lambda_+ = \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$ of the associated 'pinched quadratic-like map' is no longer connected.

Proposition 1 *For $a \in \mathcal{K} \setminus \mathcal{M}$, the set $\Lambda = \Lambda_+ \cup \Lambda_-$ is a Cantor set.*

Proof Recall that a subset $X \subset \hat{\mathbb{C}}$ is a Cantor set if it is non-empty, closed, perfect (X has no isolated points), and totally disconnected (each component of X is a single point). We consider the action of $\mathcal{F}_a = J \circ Cov_0$ as a one-to-two multifunction from D_J onto $D_1 = \mathcal{F}_a(D_J)$. As we have already observed, $\mathcal{F}_a^{-1} : D_1 \to D_J$ is a single-valued two-to-one map. Since $a \notin \mathcal{M}$, the (unique) critical value 2 of this map does not lie in $\Lambda_+ = \bigcap_{n=0}^{\infty} \mathcal{F}_a^n(D_J)$. Hence there exists some least n_0 such that $2 \notin \mathcal{F}_a^{n_0}(D_J)$. Now $\mathcal{F}_a^{n_0-1}(D_J)$ will consist of a single topological disc E and $\mathcal{F}_a^{n_0}(D_J)$ will consist of two disjoint topological discs contained in E, one of which will have the pinch point 1 on its boundary. Label these two discs E_0 and E_1, where E_1 is the disc which has 1 on its boundary. The two restrictions of $\mathcal{F}_a$,

$$f_0 = (J \circ Cov_0)|_{(E,E_0)}$$

$$f_1 = (J \circ Cov_0)|_{(E,E_1)}$$

are both homeomorphisms which are conformal on interiors. For any finite sequence $s = s_1, s_2, \ldots, s_n$, $s_i \in \{0, 1\}$, we let

$$E_s = f_{s_1} \circ f_{s_2} \circ \ldots f_{s_n}(E),$$

and for an infinite sequence $s = s_1, s_2, \ldots$, we let

$$E_s = \bigcap_{i=1}^{\infty} f_{s_1} \circ f_{s_2} \circ \ldots \circ f_{s_i}(E).$$

Note that Λ_+ is precisely the set of all points which are contained within an infinite number of images of E under mixed iteration of f_0 and f_1. In other words $\Lambda = \bigcup_{s \in S} E_s$, where S is the set of all possible infinite sequences with elements in $\{0, 1\}$.

We now show that each E_s consists of a single point. We separate the sets E_s into two distinct types.

Firstly, if s contains infinitely many 0's then E_s is contained within infinitely many images of E under mixed forward iteration of f_0 and f_1. We let A be a topological annulus, contained in $E \setminus (E_0 \cup E_1)$ and such that E_0 is surrounded by A, i.e. E_0 is contained in the bounded component of $\hat{\mathbb{C}} \setminus A$. Then E_s is surrounded by infinitely many conformal images of A, all pairwise disjoint. Thus by use of the Grötzsch inequality [6] it follows that each such E_s is a single point.

Secondly, if s contains only finitely many 0's then E_s is an image, under mixed iteration of f_0 and f_1, of $E_{111\ldots}$, the intersection of all images of E under iteration of f_1. By applying the Denjoy-Wolff Theorem [6] to $f_1 : E \to E$, we know that either f_1 has a fixed point in the interior of E or there is a fixed point of f_1 on the boundary of E to which every orbit converges. The fixed points of f_1 are of course fixed points of $\mathcal{F}_a$, and there are just four of these, counted with multiplicity, since the equation $\mathcal{F}_a(z) = z$ can be manipulated into a polynomial equation of degree four. Moreover the fixed point 1 of f_1 has multiplicity two (being parabolic), the branch f_0 of $\mathcal{F}_a$ has a fixed point $\zeta \in E_0$

(since $E_0 = f_0(E)$ has closure contained in E), and since $J(\zeta)$ is also a fixed point of $\mathcal{F}_a$ (being a fixed point of $\mathcal{F}_a^{-1}$) we have now used up all four fixed points of $\mathcal{F}_a$. Therefore f_1 can have no fixed point in the interior of E. The Denjoy-Wolff Theorem now tells us that all orbits of f_1 on E must converge to 1. Moreover 1 cannot be isolated, since every point of $\Lambda_+ \cap E$ has orbit under f_1 accumulating at 1.

That Λ_+ is perfect follows from the fact that for every infinite sequence $s = s_1, s_2, \ldots$, every set in the sequence $E_{s_1}, E_{s_1,s_2}, E_{s_1,s_2,s_3}, \ldots$ contains infinitely many points of Λ_+ (in the case of the point 1 this follows from the fact that every point of $\Lambda_+ \setminus \{1\}$ has orbit under f_1 accumulating at 1). That Λ_+ is closed follows immediately from the fact that it is the complement of Ω in D_J, together with the point 1 on the boundary of D_J.

Thus Λ_+ is a Cantor set. By symmetry $J(\Lambda_+)$ is also a Cantor set. The union of two Cantor sets contained in disjoint open discs, except for a single point in common on the boundaries of these discs, is again a Cantor set. So $\Lambda = \Lambda_+ \cup J(\Lambda_+)$ is a Cantor set. $\Box$

Remark When $a \in \mathcal{M}$, similar reasoning to that in the proof above shows that every $\mathcal{F}_a^n(D_J)$ is connected, and hence that Λ_+ is connected.

It follows from Proposition 1 that for $a \in \mathcal{K} \setminus \mathcal{M}$ the action of $\mathcal{F}_a$ on Ω can no longer be conjugate to that of the modular group on the complex upper half-plane, since Ω is no longer simply-connected. Nevertheless the action of $\mathcal{F}_a$ on Ω remains 'discontinuous', in the sense that the space of grand orbits on Ω has the structure of an orbifold. As will be shown in [5] and a future article, apart from a countable set of isolated parameter values where the singular points 2 and -2 of Cov_0^Q lie on the same grand orbit of $\mathcal{F}_a$, all the correspondences $\mathcal{F}_a$ with $a \in \mathcal{K} \setminus \mathcal{M}$ lie in a single quasi-conformal conjugacy class, and can be obtained from one another by deformations of the complex structure on the orbifold $\mathcal{O} = \Omega/\langle \mathcal{F}_a \rangle$. In the final section of this paper we investigate an example where one can follow a ray in deformation space all the way to the boundary.

We begin by describing the dynamics of a *generic* correspondence, that is to say an $\mathcal{F}_a$ with $a \in \mathcal{K} \setminus \mathcal{M}$ which does not have any critical coincidences. For example any value of a in the real interval $(-1, +1)$ will do. We pick some a in this interval as the 'base point' of our parameter space and denote it by a_*. We let H_1 denote the image of $[2, \infty]$ under Cov_0 and let H_2 be the image of $[-\infty, -2]$. The region bounded by H_1 and H_2 is a fundamental domain for Cov, and the Klein Combination Theorem tells us that the region Δ_1 bounded by H_1, H_2, and the circle through $a(= a_*)$ and 1 centred on the real axis, is a fundamental domain for the action of $J \circ Cov_0$ on the union of its images. This fundamental domain and its two images Δ_2 and Δ_3 under Cov_0 are shown in Figure 4. Write Δ for the union of $\Delta_1, \Delta_2, \Delta_3$ together with the parts $[-\infty, -2)$, $[2, \infty]$, H_1 and $H_2 \setminus \{1\}$ of their boundaries which meet only finitely many images of Δ_1. The complement of Δ consists of closed discs D_1, D_2, and D_3, where D_1 is contained in the region between H_1 and H_2 in the figure, and D_2 and

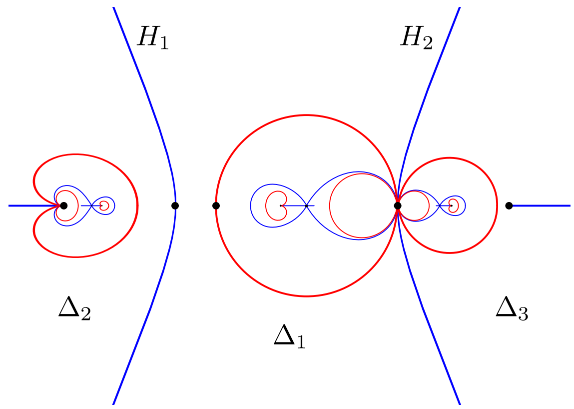

Figure 4: The fundamental domain Δ_1 and its first few images under combinations of Cov_0 and J. (The points marked by black dots on the real axis are: $-2, -1, a_*, 1$ and 2.)

D_3 are contained in left-hand and right-hand regions respectively. Now $J(\Delta)$ is contained in D_1 and $Cov_0 \circ J(\Delta)$ consists of two components, one contained in D_2 and the other in D_3. Continued iteration of J and Cov_0 gives us an infinite collection of images of Δ meeting along pairwise common edges. Each of these 'tiles' is a simply-connected set with the real axis as a mirror-symmetry line, so the set Ω, made up of the union of these images together with their common edges, has as its complement a Cantor set contained in the real line.

The orbifold $\mathcal{O}$, the grand orbit space of Ω under the correspondence, is the quotient of Δ_1 under the boundary identifications indicated in Figure 5. It is a sphere with four cone points of types $\pi, \pi, 2\pi/3$ and π/∞ (the last one being a puncture point). One way to deform the complex structure on $\mathcal{O}$ is to contract a geodesic, for example a geodesic arc from the cone point -1 to the cone point a. Choices for such an arc correspond to rationals p/q with q odd, as follows. If we 'de-identify' the point 1 on the boundary of Δ_1 then we have a (hyperbolic) rectangle. We label two of the sides l_1 and l_2 as shown in the diagram, and we say that a geodesic from -1 to a has *slope* p/q, and denote it by $\gamma_{p/q}$, if it intersects l_1 and l_2 in p points and q points respectively: the example $\gamma_{1/3}$ is illustrated in Figure 5. The geodesic $\gamma_{p/q}$ lifts to a *lamination* $\Gamma_{p/q}$ on Ω: we simply mark the same 'pattern' on each 'tile'.

The global lamination corresponding to $\gamma_{1/3}$ is illustrated in Figure 6. It is tempting to try to deform $\mathcal{F}_{a_*}$ to a correspondence $\mathcal{F}_a$ with a on the boundary of $\mathcal{K}$, by contracting the leaves of $\Gamma_{p/q}$ to points, but there are technical difficulties. Even once one has excluded the possibility of a topological obstruction arising from a pair of leaves with the same end points, it is a daunting technical task to construct isotopies shrinking unions of arcs to points in such a way that the resulting correspondence is holomorphic (see [2]). We shall discuss this

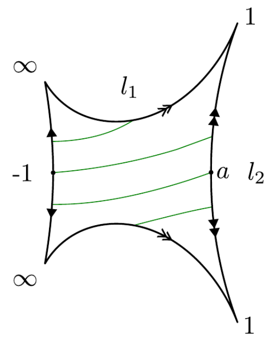

Figure 5: Δ_1 marked with its boundary identifications and the geodesic $\gamma_{1/3}$.

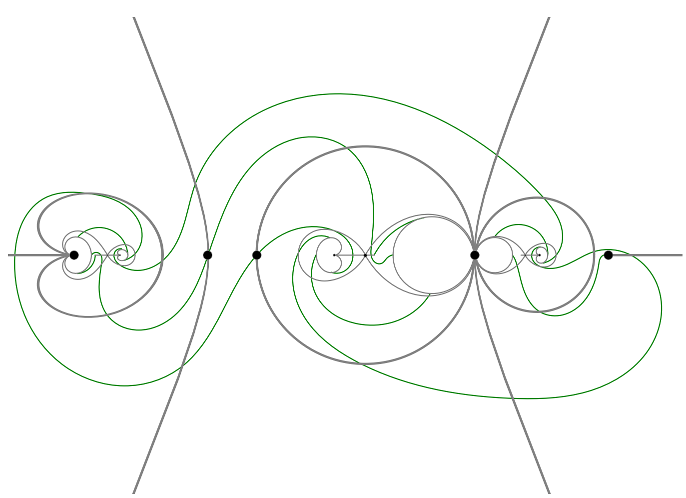

Figure 6: The global lamination $\Gamma_{1/3}$.

approach to the structure of the boundary of $\mathcal{K}$ elsewhere. Here we content ourselves with showing explicitly that the $\gamma_{1/3}$ pinch point can be reached, and describing the behaviour of the correspondence there. The way that we shall do this is by identifying a unique candidate for the parameter value where the pinched dynamics could occur.

4 An example on the boundary: Penrose point

It is apparent from Figure 6 that the leaf of $\Gamma_{1/3}$ which passes through -1 also passes through one of the two points $J \circ Cov_0(a)$. Thus if we can pinch the geodesic $\gamma_{1/3}$ on the orbifold $\mathcal{O}$, then this will happen at a value of a where $-1 \in J \circ Cov_0(a)$. But then we will also have $a \in Cov_0(J(-1))$ and so, since a is fixed by J, we will now have both $-1 \in \mathcal{F}_a(a)$ and $a \in \mathcal{F}_a(J(-1))$. Since $J(-1) \in \mathcal{F}_a(-1)$ for any value of a, we see that the three points $a, -1, J(-1)$ will become a 3-cycle.

We next note that $J(-1) \in Cov_0(a)$ if and only if

$$a^2 + aJ(-1) + J(-1)^2 = 3.$$

But

$$J(z) = \frac{(a+1)z - 2a}{2z - (a+1)}$$

so

$$J(-1) = \frac{1+3a}{3+a}.$$

Thus

$$(a^2-3)(3+a)^2 + a(1+3a)(3+a) + (1+3a)^2 = 0,$$

which simplifies to

$$a^4 + 9a^3 + 25a^2 - 9a - 26 = 0,$$

and thence to

$$(a^2 + 9a + 26)(a^2 - 1).$$

We deduce that

$$a = -\frac{9}{2} + \frac{\sqrt{23}}{2}i$$

or its complex conjugate, and we denote the value which has positive imaginary part by $a_{1/3}$. (The value with negative imaginary part will correspond to pinching $\gamma_{-1/3}$.)

Figure 7 illustrates the dynamics of $\mathcal{F}_a$ for $a = a_{1/3}$. Shown in red are the images of the circular arc which runs from -1 through $+1$ to $J(-1)$. Shown in black is the grand orbit of the point $+1$ under all branches, forward and back, of the iterated correspondence $\mathcal{F}_a$. An equivalent description of the black set is that it is the set of all images of the point $+1$ under finite 'words' made up of the symbols J and Cov_0.

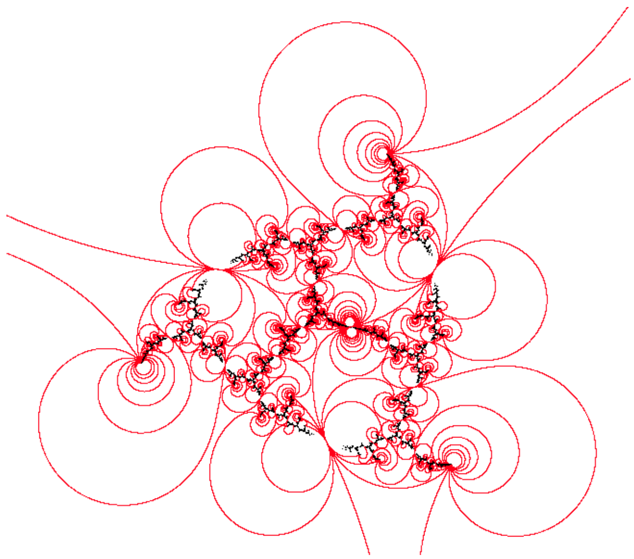

Figure 7: The correspondence $\mathcal{F}_{a_{1/3}}$. The complement Ω of the limit set is tiled by 'ideal hexagons' (Theorem 1).

Remark We have assigned $a_{1/3}$ the name 'Penrose point' in honour of Chris Penrose who originally found this example and plotted the limit set in the early 1990's (unpublished). Our re-discovery of it, and the analysis of its Fatou and limit sets reported on here, provided a key impetus for the formulation of conjectures (which we shall present elsewhere) describing the structure of the boundary of the Klein combination locus $\mathcal{K}$. 'Penrose point' lies at the tip of a promontory on the 'outer shoreline' of $\mathcal{K}$.

Theorem 1 *The correspondence $\mathcal{F}_{a_{1/3}}$ is obtained from $\mathcal{F}_{a_*}$ by pinching the geodesic $\gamma_{1/3}$ on the orbifold $\Omega(\mathcal{F}_{a_*})/\langle\mathcal{F}_{a_*}\rangle$. Moreover:*

1. *The regular set $\Omega = \Omega(\mathcal{F}_{a_{1/3}})$ of $\mathcal{F}_{a_{1/3}}$ is tessellated into ideal hexagons by the images of the circular arc from -1 through $+1$ to $J(-1)$. The action of $\mathcal{F}_{a_{1/3}}$ on Ω is a faithful action of the free product of the cyclic group $\{Id, J\}$ with the $(3:3)$ correspondence Cov (where $Cov_0 = Cov \setminus \{Id\}$ acts on the hexagon containing ∞ as a pair of rotations fixing ∞).*

2. *Ω has four connected components Ω_k, $k = 1, 2, 3, 4$, each conformally homeomorphic to the (open) upper half-plane.*

3. *For each of $k = 1, 2, 3, 4$ the action of the set of branches of iterates of the correspondence $\mathcal{F}_{a_{1/3}}$ which stabilise Ω_k is conformally conjugate to the action of a free product of groups $C_3 * C_\infty$ on the upper half-plane, where C_3 is generated by an elliptic Möbius transformation of order three and C_∞ is generated by a parabolic Möbius transformation.*

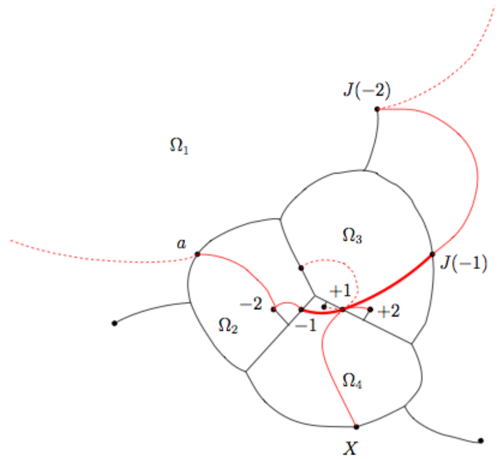

Figure 8: Schematic picture of the components Ω_i of Ω, the positions of key points, the arc $L \cup J(L)$ (in thick red) and its images under Cov_0 (in thin red) and $J \circ Cov_0$ (in dashed red). The incomplete dashed arcs at the top and left of the picture are in reality joined by a large loop (see the computer plot on the left in Figure 9). The two images of a under Cov_0 are $J(-1)$ and X.

Proof The positions of the key points we shall need in the proof are indicated in Figure 8. The circular arc referred to in the statement of the theorem is made up of an arc L from -1 to $+1$, and its image $J(L)$ from $+1$ to $J(-1)$. Consider the images of $L \cup J(L)$ under the identity, Cov_0, and $J \circ Cov_0$. These are shown in Figure 8 and the left hand plot of Figure 9 . We assert that $L \cup J(L)$, together with its image under Cov_0 running from -1 to $a = a_{1/3}$ (through a cusp at -2), and its image under $J \circ Cov_0$ running from a to $J(-1)$ (through a cusp at $J(-2)$), form a piecewise smooth simple closed curve, invariant under J. Following the same notation as in earlier sections of this paper, denote the component of the complement of this curve containing ∞ by Δ_J and the closure of the other component by D_J. We further assert that D_J and its images under Cov_0 (plotted on the right in Figure 9) have disjoint interiors, that they are pairwise contiguous along L, $J(L)$ and the branch of $Cov_0(J(L))$ emanating from $+1$, and that the union of D_J with its two images is therefore again a topological disc bounded by a piecewise smooth curve.

These assertions can be proved as follows (we omit details). Firstly, local analysis around the points of the period three cycle (which is parabolic), and around the (also parabolic) fixed point $+1$ and its image -2 under Cov_0, can be used to verify that the intersections of the arcs with neighbourhoods of the end points of L and their images are arranged as shown. Away from these neighbourhoods the arcs and their images are a definite distance apart and numerical estimation

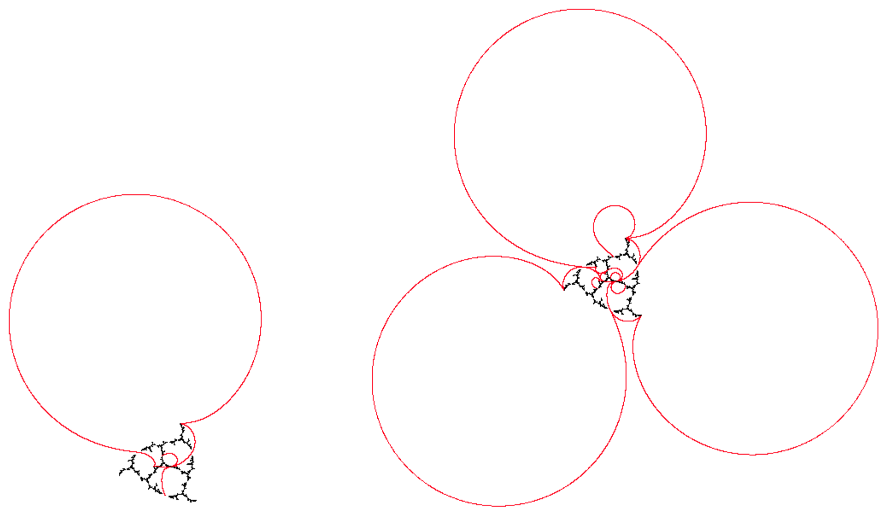

Figure 9: Computer plots. Left: $L \cup J(L)$ together with its images under Cov_0 and $J \circ Cov_0$. Right: this set together with its image under a further application of Cov_0.

can be used complete the proof of the assertions. The same methods, of local analysis around the period three cycle and numerical estimation away from it, can be used to prove that the cycle can be 'unpinched' by a suitable small perturbation of the parameter value $a_{1/3}$ (see the first remark following this proof), and thus that $\mathcal{F}_{a_{1/3}}$ is the correspondence obtained from $\mathcal{F}_{a_*}$ by pinching $\gamma_{1/3}$.

We deduce from the assertions above that D_J can be extended as follows, to become a fundamental domain for the action of Cov on $\hat{\mathbb{C}}$. Join a to ∞ by any smooth curve M which is disjoint from its images under Cov_0 and from $\partial\Delta_J$. Then the simple closed curve made up of M, the branch of $Cov_0(M)$ running from ∞ to $J(-1)$, and segments of the boundary of D_J running fom $J(-1)$ to -1 and -1 to a, will together bound a fundamental domain Δ_{Cov} for Cov. Now observe that the complement Δ_J of D_J is a fundamental domain for J, and that

$$\overline{\Delta}_{Cov} \cup \overline{\Delta}_J = \hat{\mathbb{C}}.$$

This is precisely the condition we need to apply the 'ping-pong principle'. The Klein Combination Theorem has a simple statement in the case that the two fundamental domains concerned have disjoint boundaries, or when these boundaries meet at a single point (as in Section 2) but one has to take care when the boundaries meet along arcs, as in our situation here. The intersection $\Delta = \Delta_J \cap \Delta_{Cov}$ is the interior of a triangle which has two vertices of angle zero and one vertex of angle $2\pi/3$ (the vertex at ∞). Let $\mathcal{H}$ denote the ideal hexagon formed by the union of Δ with its two images under Cov_0, together with the points which lie

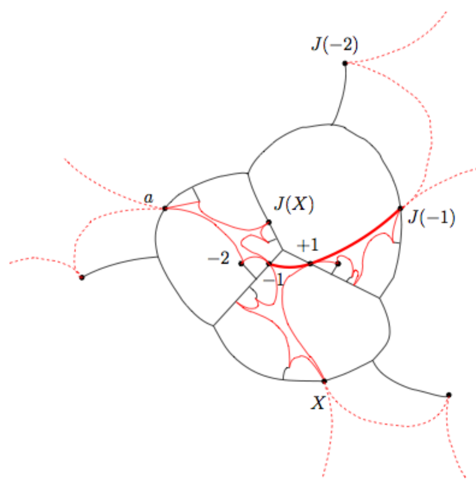

Figure 10: Sketch of the positions of the four ideal hexagons $\mathcal{H}_i$. (The dashed lines are the boundary of $\mathcal{H}_1$, which contains ∞.)

on the boundaries of two of these triangles, and ∞ (which lies on the boundary of all three). Thus $\mathcal{H}$ is the external region (containing ∞) in the plot on the right in Figure 9. The ping-pong principle tells us at once that the free product $\{Id, J\} * Cov$ acts faithfully on the union of images of $int(\mathcal{H})$, but we can do better than this and include the edges of $\mathcal{H}$ (though not its vertices). To see this, consider the following four ideal hexagons, which are disjoint apart from certain of their vertices: $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = J(\mathcal{H})$, and the two images of $\mathcal{H}_2$ under Cov_0, which we denote $\mathcal{H}_3$ and $\mathcal{H}_4$.

These four hexagons $\mathcal{H}_i$ are sketched in Figure 10 and can also be identified on the computer plot Figure 7. It is easily checked that any application of J or a branch of Cov_0 to any of these $\mathcal{H}_i$ takes it either to another of the $\mathcal{H}_i$ or to a hexagon that has an edge in common with one of them. Setting Ω to be the union of the images of $\mathcal{H}$ together with its edges (but not its vertices) we deduce that $F_{a_{1/3}}$ has a proper discontinuous action on Ω, that this is a faithful action of the free product of $\{Id, J\}$ with Cov, and that the 'centres' of the hexagons (the images of the point $\infty \in \mathcal{H}$), each have stabiliser a conjugate of the correspondence Cov. This gives us Statement 1 of the Theorem. Furthermore, the components of Ω are built up inductively from the four $\mathcal{H}_i$ by adjoining ideal hexagons along edges. Thus Ω is the disjoint union of four components, each containing one of the $\mathcal{H}_i$, and each homeomorphic to a disc (Statement 2).

The way that the four components of Ω are mapped to one another by Cov_0 and J is determined by where the initial four ideal hexagons $\mathcal{H}_i$ are mapped. Thus Cov_0 stabilises Ω_1 and sends each of $\Omega_2, \Omega_3, \Omega_4$ to the other two, and J exchanges

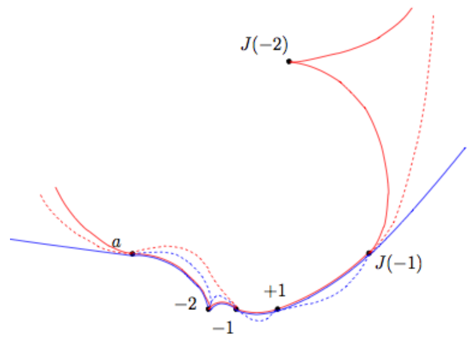

Figure 11: The boundaries of Δ_J (solid red) and Δ_{Cov} (solid blue) coincide along the segment between a and $J(-1)$. We obtain new fundamental domains Δ'_J and Δ'_{Cov} by modifying parts of these boundaries to the positions indicated by the dashed red and blue lines respectively.

Ω_1 with Ω_2 and also exchanges Ω_3 with Ω_4 (see Figure 8 for the labelling of the components). It is easily checked that Ω_2 is stabilised by $J \circ Cov_0 \circ J$ (which is a pair of rotations of Ω_2 of order three, inverse to one another) and by the two branches of $Cov_0 \circ J \circ Cov_0$ (inverse to one another) which fix the point -2 on the boundary of Ω_2. Furthermore, since the images of $\mathcal{H}_2$ under combinations of these elliptic and parabolic transformations fully tile Ω_2, we deduce that they freely generate the set of branches of iterates of $\mathcal{F}_{a_{1/3}}$ which stabilise Ω_2, and that this set of branches forms a group isomorphic to $C_3 * C_\infty$. $\square$

Remarks

1. To show that the parabolic 3-cycle can be 'unpinched' to deform the correspondence into one which satisfies the generic set-up of Definition 2, we first keep the value of a fixed at $a_{1/3}$ and modify Δ_J and Δ_{Cov} to new fundamental domains Δ'_J and Δ'_{Cov} for J and Cov, in such a way that we still have $\overline{\Delta'}_{Cov} \cup \overline{\Delta}'_J = \hat{\mathbb{C}}$ but now the boundaries of Δ'_J and Δ'_{Cov} meet at just four points, namely the point $+1$ and the points of the 3-cycle.
We illustrate how this can be done in Figure 11 (one has to cut thin strips off certain edges of Δ_J and Δ_{Cov}, and glue them onto other edges). If one now makes a sufficiently small perturbation of the parameter a, in a direction which splits the (parabolic) 3-cycle into a pair of 3-cycles, one attracting and the other repelling, then Δ'_J and Δ'_{Cov} will move apart so that their boundaries only meet at $+1$.

2. The limit set Λ of $\mathcal{F}_{a_{1/3}}$ (the complement of Ω) can be described combinatorially as the quotient of the unpinched limit set (a Cantor set) under the identifications induced by contracting the leaves of $\Gamma_{1/3}$ to points. Alternatively, we may view $\mathcal{F}_{a_{1/3}}$ on $\hat{\mathbb{C}}$ as a kind of mating of four copies of an action of $C_3 * C_\infty$ on the unit disc, glued together along their boundaries, and then Λ becomes a

quotient space of the union of the four boundary circles of these discs.

3. In this article we have considered just one example of a correspondence at the boundary of the Klein combination locus. At other boundary points the behaviour can be very different. Further examples, and conjectures concerning the overall structure of the bounary of the Klein combination locus, will be presented elsewhere.

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