LTCC

Holomorphic Dynamics and Hyperbolic Geometry
(February-March 2013)

Week 1 Exercises

1. For the angle-doubling map $t \to 2t \bmod 1$ on the circle $\mathbb{R}/\mathbb{Z}$ prove that the periodic points are the points $t \in [0, 1)$ of the form $t = m/(2^n - 1)$ (where $0 \leq m < 2^n - 1$ with $m, n \in \mathbb{N}$).

2. Show that $h : z \to z + 1/z$ is a semiconjugacy from $f : z \to z^2$ to $g : z \to z^2 - 2$ (that is, h is a surjection satisfying $hf = gh$) and that h sends the Julia set of f (the unit circle) onto the real interval $[-2, +2]$.

3. Find a Möbius transformation which sends the upper half plane $\mathcal{H}_+$ bijectively onto the unit disc $\mathbb{D}$. Assuming the structure of $Aut(\mathbb{D})$ (Prop 2.9) prove that $Aut(\mathcal{H}_+) = PSL(2, \mathbb{R})$ (Cor 2.10).

4. Let $w = e^{i\theta}(z - a)/(1 - \bar{a}z)$ with $\theta \in \mathbb{R}$ and a in the open unit disc $\mathbb{D}$. Show that $\left|\frac{dw}{dz}\right| = \frac{1-|w|^2}{1-|z|^2}$ and hence $\frac{2|dz|}{1-|z|^2} = \frac{2|dw|}{1-|w|^2}$. Deduce that the infinitesimal metric $d\rho = \frac{2|dz|}{1-|z|^2}$ is invariant under $Aut(\mathbb{D})$.
(To verify that $d\rho$ is what we get when we transfer the Poincaré metric from the upper half-plane to $\mathbb{D}$, it now suffices to check that integrating $d\rho$ gives the distance between 0 and $t \in \mathbb{D} \cap \mathbb{R}$ to be $\ln |(0, t; -1, +1)|$.)

5. Show that a rational map f OF DEGREE > 1 is conjugate to a polynomial of the form $z \to z^n$ (SOME $n > 1$) if and only if there exist distinct points $z_0, z_1 \in \hat{\mathbb{C}}$ such that $f^{-1}(z_0) = \{z_0\}$ and $f^{-1}(z_1) = \{z_1\}$.

6. Show that every degree 2 polynomial $z \to \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$) is conjugate to a (unique) one of the form $z \to z^2 + c$.

7. Let f be the rational map

$$z \to \frac{-2z - 1}{z^2 + 4z + 2}$$

Find the critical points of f and their orbits. Deduce that f is conjugate to $z \to z^2 - 1$.

SB 18/2/13, corrected 21/2/13