Holomorphic Dynamics and Hyperbolic Geometry

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Overview

Rational maps are self-maps of the Riemann sphere of the form $z \to p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials. Kleinian groups are discrete subgroups of $PSL(2, \mathbb{C})$, acting as isometries of 3-dimensional hyperbolic space and as conformal automorphisms of its boundary, the Riemann sphere. Both theories experienced remarkable advances in the last two decades of the 20th century and are very active areas of continuing research. The aim of the course is to introduce some of the main techniques and results in the two areas, emphasising the strong connections and parallels between them.

Topics to be covered in 5 two-hour lectures

(This list may change as the course progresses, depending on the interests of the group)

1. Dynamics of rational maps: The Riemann sphere and rational maps (basic essentials from complex analysis); conformal automorphisms of the sphere, plane and disc; Schwarz's Lemma; the Poincaré metric on the upper half-plane and unit disc; conjugacies, fixed points and periodic orbits (basic essentials from dynamical systems); spherical metric; equicontinuity; Fatou and Julia sets (definition).

2. Fatou and Julia sets: Normal families and Montel's Theorem; characterisations and proporties of Fatou and Julia sets; types of Fatou component; linearization theorems (Koenigs, Böttcher, Siegel, Brjuno, Yoccoz).

3. Hyperbolic geometry and Kleinian groups: Hyperbolic 3-space and its isometry group; Kleinian groups; ordinary sets and limit sets; fundamental domains, Poincaré's polyhedron theorem; examples of Fuchsian and Kleinian groups and their limit sets.

4. Quadratic maps and the Mandelbrot set: The Mandelbrot set and its connectivity; geography of the Mandelbrot set: internal and external rays; introduction to kneading theory (Milnor-Thurston); open questions.

5. Further topics (selection from the following): The Measurable Riemann Mapping Theorem and its applications to holomorphic dynamics and Kleinian groups; polynomial-like mappings and renormalisation theory; Thurston's Theorem (characterizing topological branched-covering maps equivalent to rational maps); conformal surgery, matings; the 'Sullivan Dictionary' between holomorphic dynamics and Kleinian groups.

Prerequisites

Undergraduate complex analysis, linear algebra and elementary group theory.

1 Introduction

1.1 Overview

The objective of these lecture notes is to develop some of the main themes in the study of iterated *rational maps*, that is to say maps of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ to itself of the form

$$z \to \frac{p(z)}{q(z)}$$

(where p and q are polynomials with complex coefficients), and the study of *Kleinian groups*, discrete groups of maps from $\hat{\mathbb{C}}$ to itself, each of the form

$$z \to \frac{az+b}{cz+d}$$

(where a, b, c, d are complex numbers, with $ad - bc \neq 0$).

We shall develop these studies in parallel: although there is no single unifying theory encompassing both areas there are tantalizing similarities between them, and results in one field frequently suggest what we should look for in the other (*Sullivan's Dictionary*).

The study of iterated rational maps had its first great flowering with the the work of the French mathematicians Julia and Fatou around 1918-20, though its origins perhaps lie earlier, in the late 19th century, in the more geometric work of Schottky, Poincaré, Fricke and Klein. It has had its second great flowering over the last 30 years, motivated partly by the spectacular computer pictures which started to appear from about 1980 onwards, partly by the explosive growth in the subject of chaotic dynamics which started about the same time, and not least by the revolutionary work in three-dimensional hyperbolic geometry initiated by Thurston in the early 1980's. In the intervening period Siegel (in the 1940s) had proved key results concerning local linearisability of holomorphic maps, and Ahlfors and Bers (in the 1960s) following pioneering work of Teichmüller (in the late 1930s) had developed quasiconformal deformation theory for Kleinian groups: the stage was set for an explosion of interest, both experimental and analytical. Some of the names associated with this second great wave of activity are Mandelbrot, Douady, Hubbard, Sullivan, Herman, Milnor, Thurston, Yoccoz, McMullen and Lyubich. Both subjects are still very active indeed: as we shall see, some of the major conjectures are still waiting to be proved. But the remarkable mixture of complex analysis, hyperbolic geometry and symbolic dynamics that constitutes the subject of holomorphic dynamics yields powerful methods for problems which at first sight might appear only to concern only real mathematics. For example the most conceptual proof of the universality of the Feigenbaum ratios for period doubling renormalisation of real unimodal maps is that of Sullivan (1992) using complex analysis.

We start our study of rational maps and Kleinian groups - as we mean to go on - with motivating examples.

1.2 The family of maps $z \to z^2 + c$

(i) $c = 0$

Here the dynamical behaviour is straightforward. When we iterate $z \to z^2$ any orbit started inside the unit circle heads towards the point 0, any orbit started outside the unit circle heads towards ∞, and any orbit started on the unit circle remains there. The two components of $\{z : |z| \neq 1\}$ are known as the *Fatou set* of the map and the circle $|z| = 1$ is called the *Julia set*. On the unit circle itself the dynamics are those of the *shift*, namely if we parametrise the circle by $t \in [0, 1) \subset \mathbb{R}$ $\quad (t = arg(z)/2\pi)$: then $z \to z^2$ sends $t \to 2t$ mod 1.

Any $t \in [0, 1)$ of the form $t = m/(2^n - 1)$ (for $0 \leq m < 2^n - 1$ integer) is periodic, of period n (exercise: prove this). Hence the periodic points form a dense set on the unit circle. Moreover the map $z \to z^2$ has *sensitive dependence on initial condition*, since the map on the unit circle doubles distance.

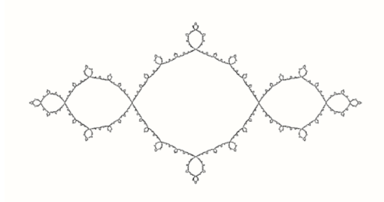

Figure 1: Julia set for $z \to z^2 - 1$

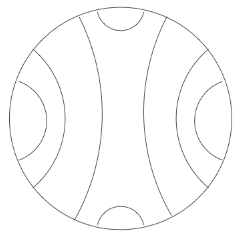

Figure 2: Lamination for $z \to z^2 - 1$ (first few leaves).

(ii) $c = -1$

When we vary c just a little from 0 the dynamical picture remains like that for $z \to z^2$. There is a single attractive fixed point (but this is no longer 0 itself), the Fatou set is a pair of (topological) discs, the basins of attraction of the finite fixed point and ∞ respectively, and the Julia is a (fractal) topological circle separating these discs. However as $|c|$ becomes larger the Julia set becomes more and more distorted and eventually self-intersects. For example once c has reached -1 the dynamical behaviour is rather more complicated to describe (see Figure 1). The Fatou set now has infinitely many components. There is a fixed point at ∞ to which every orbit started in the component of the Fatou set outside the 'filled Julia set' is attracted, and a period 2 cycle $0 \to -1 \to 0 \to -1 \to ...$ towards which every orbit started in any other component of the Fatou set is attracted. An orbit which starts on the common boundary of the two attractors (the 'Julia set', which we shall define formally soon) remains on that boundary. Combinatorially, the Julia set in this example is a *quotient* of the circle, and the dynamics are those of the corresponding *quotient* of the shift. Figure 2 shows the first few identifications on the unit circle in the construction of this quotient: contracting the leaves on the closed unit disc gives a model of the *filled Julia set* for $z \to z^2 - 1$.

(iii) $c = i$

See Figure 3. Note that the point 0 is *preperiodic* for this map ($0 \to i \to -1 + i \to -i \to -1 + i ...$). It can be

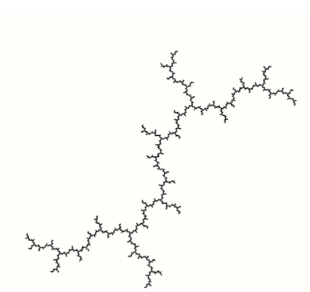

Figure 3: Julia set for $z \to z^2 + i$

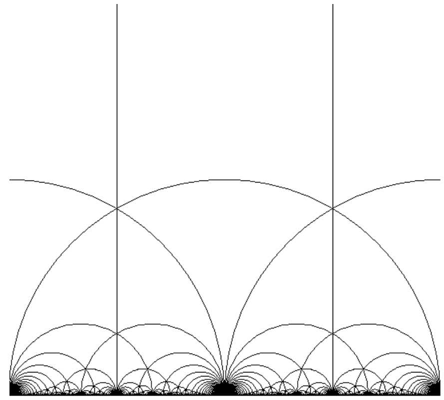

Figure 4: The modular group action on the upper half-plane

proved that whenever c is such that the critical point 0 of $z \to z^2 + c$ is preperiodic but not periodic, the Julia set is a *dendrite* (that is a connected, simply-connected set with empty interior).

(iv) $c = -2$

Here again 0 is preperiodic, and the dendrite (not drawn here) is the real interval $[-2, 2]$.

Exercise Show that $h : z \to z + 1/z$ is a semiconjugacy from $f : z \to z^2$ to $g : z \to z^2 - 2$ (that is, h is a surjection satisfying $hf = gh$) and that h sends the Julia set of f (the unit circle) onto the real interval $[-2, +2]$.

For $|c|$ sufficiently large the Julia set becomes disconnected - in fact it becomes a Cantor set. The set of all values of $c \in \mathbb{C}$ such the Julia set is connected is known as the *Mandelbrot Set*.

1.3 The modular group $PSL(2, \mathbb{Z})$

The *modular group* $PSL(2, \mathbb{Z})$ is the group of Möbius transformations

$$z \to \frac{az + b}{cz + d}$$

such that a, b, c, d are *integers* with $ad - bc = 1$. It is easy to see that $PSL(2, \mathbb{Z})$ maps the open upper half $\mathcal{H}_+$ of the complex plane to itself, the open lower half plane $\mathcal{H}_-$ to itself and the extended real axis $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ to itself (see Figure 4).

We remark that the modular group is generated by $S : z \to -1/z$ and $T : z \to z+1$. All relations in the group are consequences of the pair of relations $S^2 = I$, $(ST)^3 = I$. The region $\Delta = \{z : |z| \leq 1, Re(z) \leq 1/2, Im(z) > 0\}$ is a *fundamental domain* for the action of $PSL(2, \mathbb{Z})$ on the upper half plane: $\mathcal{H}_+$ is 'tiled' by the translates of Δ under elements of the group. Similarly $\mathcal{H}_-$ is tiled by the mirror image of Δ and its translates. Both sets of tiles accumulate on $\hat{\mathbb{R}}$. Just as is the case for rational maps, the action of a Kleinian group G partitions the Riemann sphere into two disjoint completely invariant subsets, an *ordinary set* $\Omega(G)$ (in the case of the modular group this is $\mathcal{H}_+ \cup \mathcal{H}_-$), and a *limit set* $\Lambda(G)$ (in this case $\bar{\mathbb{R}}$) on which the system exhibits *sensitive dependence on initial conditions*: arbitrarily close to any point in $\Lambda(G)$ we can find another point and an element of G sending the two points arbitrarily far apart.

2 Dynamics of rational maps

2.1 The Riemann sphere

The *extended complex plane* is $\mathbb{C}$ together with an extra point '∞'. The topology on $\mathbb{C} \cup \{\infty\}$ can be described as follows. Let S^2 denote the unit sphere in $\mathbb{R}^3$, regard $\mathbb{C}$ as the plane $\mathbb{R}^2 \subset \mathbb{R}^3$ (which cuts through S^2 at its equator), and let $N = (0,0,1)$ denote the 'north pole' of S^2. Stereographic projection from N defines a homeomorphism $\pi : S^2 - \{N\} \to \mathbb{C}$. Extending π to send N to ∞ we obtain a homeomorphism from S^2 to $\mathbb{C} \cup \infty$, where the latter is topologised by taking as neighbourhoods of ∞ the sets $\{z : |z| > R\} \cup \infty$. However we need more than just a *topology* on $\mathbb{C} \cup \infty$: we give S^2 the structure of a *Riemann surface* by equipping it with charts (homeomorphisms) $\phi_1 : \mathbb{C} \to S^2 - \{N\}$ and $\phi_2 : \mathbb{C} \to S^2 - \{S\}$ such that $\phi_2^{-1}\phi_1$ is an analytic bijection on the overlap. We may take ϕ_1 to be the inverse π^{-1} of stereographic projection from the north pole and ϕ_2 to be the inverse of sterographic projection from the south pole, followed by complex conjugation. The overlap $\phi_2^{-1}\phi_1$ is then $z \to \bar{z}/|z|^2 = 1/z$.

Equivalently we can put a complex structure on $\mathbb{C} \cup \infty$ by regarding it as the *complex projective line*

$$\mathbb{CP}^1 = \{\mathbb{C}^2 - (0,0)\}/\mathcal{R}$$

where $\mathcal{R}$ is the relation $(z, w) \sim (\lambda z, \lambda w)$ for $\lambda \in \mathbb{C} - 0$. An equivalence class $[z, w]$ contains $(z/w, 1)$ if $w \neq 0$ or $(1, w/z)$ if $z \neq 0$, so we may think of $\mathbb{CP}^1$ as the union of two copies of the complex plane glued together, $\mathbb{C}_1 \cup \mathbb{C}_2/(z_1 \sim 1/z_2)$. The bijection

$$\mathbb{CP}^1 \leftrightarrow \hat{\mathbb{C}}$$

is given by $[z, w] \leftrightarrow z/w$ when $w \neq 0$ and $[z, 0] \leftrightarrow \infty$. We shall use the term *Riemann sphere* interchangeably for $\hat{\mathbb{C}}$ or $\mathbb{CP}^1$, but we shall tend to use the notation $z \in \hat{\mathbb{C}}$ rather than $[z, w]$ for an individual point, just for convenience: all polynomial expressions in the former form can if necessary be re-written in the latter form simply by introducing a homogenising variable w.

2.2 Basic essentials from complex analysis

Definitions An open connected set $\Omega \subset \mathbb{C}$ is called a *domain*.

$f : \Omega \to \mathbb{C}$ is said to be *differentiable* at $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

$f : \Omega \to \mathbb{C}$ is said to be *holomorphic* if f is differentiable at all $z_0 \in \Omega$.

Theorem 2.1 *Let f be holomorphic on the domain $\Omega \subset \mathbb{C}$ and let $z_0 \in \Omega$. Let R denote the radius of the largest disc which has centre z_0 and is contained in Ω. Then for all z with $|z - z_0| < R$ the Taylor series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f at z_0 converges absolutely to the value $f(z)$.*

This is a classical theorem of complex analysis. The coefficients a_n are given by the formulae

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where C is a positively-oriented circle around z_0, or equivalently by

$$a_n = \frac{f^n(z_0)}{n!}$$

A function expressible as a power series is called *analytic*. Thus Theorem 2.1 says that a holomorphic function on a domain $\Omega \subset \mathbb{C}$ is analytic. The converse is also a well known result: every function expressible as a power series is holomorphic on the disc of convergence of the series, and its derivative is given by term-by-term differentiation.

There is a geometric interpretation for the statement that a function f is differentiable at z_0. If $f'(z_0) \neq 0$, then near z_0 we have $f(z) - f(z_0) \sim f'(z_0)(z - z_0)$ so f acts on $z - z_0$ by multiplying it by the scaling factor $|f'(z_0)|$ and turning it through an angle $arg(f'(z_0))$. Thus in particular if $f'(z_0) \neq 0$ the function f is *conformal* (angle-preserving) at z_0. If $f'(z_0) = 0$, then on a small disc centred at z_0 we have $f(z) \sim f(z_0) + a_n(z - z_o)^n$ for the first coefficient $a_n \neq 0$ and f acts on this disc as an *n-to-1 branched covering map* (branched at z_0): note that f is then *not* conformal at z_0, indeed it multiplies angles at z_0 by n.

If $f : \Omega \to \mathbb{C}$ is holomorphic except at *isolated singularities* (isolated points where f is undefined or not differentiable) then we say that f is *meromorphic* if all these singularities are either *removable* or *poles*, or equivalently if for each $z_0 \in \Omega$ there is a disc neighbourhood D of z_0 such that the Laurent series for f in the punctured disc $D - \{z_0\}$ has the form $\sum_{n=-m}^{+\infty} a_n(z - z_0)^n$. Recall that z_0 is said to be a *pole of order m* if $m > 0$ is such that $a_{-m} \neq 0$ but $a_{-n} = 0$ for all $n > m$, and that z_0 is said to be *removable* if $a_{-n} = 0$ for all $n > 0$. When z_0 is a removable singularity we can set $f(z_0) = a_0$ and thereby extend f to a function differentiable at z_0, and when z_0 is a pole $\lim_{z \to z_0} f(z) = \infty$ so we can extend the definition of f by setting $f(z_0) = \infty$ and regard f as a continuous function $f : \Omega \to \hat{\mathbb{C}} = \mathbb{C} \cup \infty$. This extension is generally called *meromorphic* too. (Note that if our original $f : \Omega \to \mathbb{C}$ has any *essential singularities* there is no way to assign values at these singularities to obtain a continuous extension $f : \Omega \to \mathbb{C} \cup \infty$ since in any neighbourhood of an essential singularity f takes values arbitrarily close to any given value.)

There is a nice way to characterise a meromorphic function $f : \Omega \to \hat{\mathbb{C}}$ (Ω a domain in $\mathbb{C}$), making use of the 'duality' between '0' and '∞'. Let σ denote the function $z \to 1/z$. Then around any pole z_0 of f the function σf is analytic, since $f(z)$ has an expression as a Laurent series

$$f(z) = (z - z_0)^{-m} \sum_{n=0}^{\infty} b_n (z - z_0)^n \quad (b_0 \neq 0)$$

and taking the reciprocal of this expression we obtain for $\sigma f(z)$ a series of the form

$$\sigma f(z) = (z - z_0)^m \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where $c_0 = 1/b_0$. It follows that $f : \Omega \to \hat{\mathbb{C}}$ is meromorphic if and only if f is analytic at those points z_0 where $f(z_0) \neq \infty$ and σf is analytic at those where $f(z_0) \neq 0$.

Finally, for full generality, we allow Ω to be a domain in $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ and not just in $\mathbb{C}$ and we say that $f : \Omega \to \hat{\mathbb{C}}$ is *meromorphic at* ∞ if $f\sigma$ is meromorphic at 0. The class of functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which are meromorphic on $\mathbb{C}$ and at ∞ are precisely the functions we are interested in: they are the functions which, provided we replace f by $\sigma f, f\sigma$ or $\sigma f \sigma$ as appropriate, have a Taylor series expansion at every point of $\hat{\mathbb{C}}$.

Definition $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *holomorphic* if f is meromorphic at every point of $\mathbb{C}$ and at ∞.

2.3 Rational maps and critical points

Theorem 2.2 *$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic if and only if f is a rational function, that is to say there exist polynomials $p(z), q(z)$, with complex coefficients, such that $f(z) = p(z)/q(z)$ for all $z \in \hat{\mathbb{C}}$.*

Proof It is an elementary exercise to show that any rational map f is meromorphic both at points of $\mathbb{C}$ and at ∞, since by the Fundamental Theorem of Algebra f has the form

$$f(z) = c\frac{(z - \alpha_1)^{m_1} ... (z - \alpha_r)^{m_r}}{(z - \beta_1)^{n_1} ... (z - \beta_s)^{n_s}}$$

For the converse, let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic. Then f has finitely many poles (else $1/f$ has a convergent sequence of zeros, which, by Theorem 2.1, is only possible if $1/f$ is identically zero). Let these poles be $\beta_1, ..., \beta_s$, of order $n_1, ...n_s$ respectively. Then

$$g(z) = (z - \beta_1)^{n_1} ... (z - \beta_s)^{n_s} f(z)$$

is analytic on $\mathbb{C}$, and so g can be written in the form

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

Since f is meromorphic at ∞ so is g. Thus $g\sigma$ is meromorphic at 0. In other words $\sum_{n=0}^{\infty} a_n z^{-n}$ has a pole or a removable singularity at $z = 0$. It follows that only finitely many of the a_n are non-zero and hence g is a polynomial. QED

Comments

1. This is a very powerful result: it tells us that any holomorphic $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is determined by a *finite* set of data, for example the poles and zeros of f together with the value of f at one other point.

2. We can write a rational map $f(z) = p(z)/q(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in terms of *homogeneous coordinates* on $\mathbb{CP}^1$ as follows: write

$$p(z) = \sum_{m=0}^{d} a_m z^m; \qquad q(z) = \sum_{m=0}^{d} b_m z^m$$

(where if necessary extra zero coefficients have been added to give p and q the same degree). Now define

$$f([z, w]) = [\sum_{m=0}^{d} a_m z^m w^{d-m}, \sum_{m=0}^{d} b_m z^m w^{d-m}]$$

Let $f(z) = p(z)/q(z)$, where p and q are polynomials of degree d_p and d_q respectively, with no common zeros. Then a general point $\zeta \in \hat{\mathbb{C}}$ has $max(d_p, d_q)$ inverse images (just consider the equation $\zeta = p(z)/q(z)$, that is to say $p(z) - \zeta q(z) = 0$: this has $max(d_p, d_q)$ solutions z for any ζ in general position). We define the *degree* of f to be $max(d_p, d_q)$. Thus, for example, rational maps of degree 1 have $f(z) = p(z)/q(z)$ where $p(z) = az + b$ and $q(z) = cz + d$ (but $ad - bc \neq 0$ else p is a constant multiple of q).

Definition A *critical point* of a rational map f is a point z_0 where the degree one term of the Taylor series for f vanishes, in other words the derivative $f'(z_0)$ vanishes.

As usual we replace f by $f\sigma$ here if $z_0 = \infty$, by σf if $f(z_0) = \infty$ and by $\sigma f \sigma$ if both are ∞, so that an appropriate Taylor series exists. Looked at topologically a critical point of f is a *branch point* of f, a point z_0 such that $f(z) - f(z_0)$ has a factor $(z - z_0)^n$ for some $n > 1$, and thus in particular where $f^{-1}f(z_0)$ consists of less than d distinct points. (But for $d > 2$ it does not follow that z_0 is a critical point just because $f^{-1}f(z_0)$ consists of less than d distinct points (exercise!).) Writing $f(z) = p(z)/q(z)$, we see that $f'(z) = 0 \Leftrightarrow q'(z)p(z) - p'(z)q(z) = 0$.

Proposition 2.3 *A degree d rational map has $2d - 2$ critical points (counted with multiplicity)*

Proof In the generic case both p and q have degree d and $q'(z)p(z) - p'(z)q(z)$ is generically a polynomial of degree $2d - 2$ (since $q'(z)p(z)$ and $p'(z)q(z)$ have the same degree $2d - 1$ term). In the non-generic case we obtain the same result if we adopt the right notion of 'multiplicity': this is best proved topologically using an argument based on Euler characteristics (see later for a full proof). QED

2.4 Conformal automorphisms of $\hat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{D}$

The invertible holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are the *conformal automorphisms* of the Riemann sphere. They form a group $Aut(\hat{\mathbb{C}})$.

Proposition 2.4 *The conformal automorphisms of $\hat{\mathbb{C}}$ are the rational maps of form*

$$f(z) = \frac{az+b}{cz+d}$$

having $a, b, c, d \in \mathbb{C}$ and $ad \neq bc$.

Proof By Theorem 2.2 for f to be holomorphic it must be rational, but to be injective it must have degree 1. Conversely, any f of this form is invertible since it has inverse $f^{-1}(z) = (dz-b)/(-cz+a)$. QED

Maps of the form $f(z) = (az+b)/(cz+d)$ having $a, b, c, d \in \mathbb{C}$ and $ad \neq bc$ are called *fractional linear* or *Möbius* transformations.

Properties of Möbius transformations

1. Any invertible linear map $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ has the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cw+dw \end{pmatrix}$$

and passes to a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ which in our coordinate z/w on $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is

$$z/w \to \frac{az+bw}{cz+dw} = \frac{az/w+b}{cz/w+d}$$

(where $(a\infty+b)/(c\infty+d)$ is to be interpreted as a/c and so on).

2. Composition of linear maps passes to composition of Möbius transformations. The group of all Möbius transformations is therefore

$$PGL(2,\mathbb{C}) = \frac{GL(2,\mathbb{C})}{\{\lambda I; \lambda \in \mathbb{C} - \{0\}\}} = \frac{SL(2,\mathbb{C})}{\{\pm I\}} = PSL(2,\mathbb{C})$$

where $GL(2,\mathbb{C})$ denotes the group of all invertible 2×2 matrices and $SL(2,\mathbb{C})$ denotes those of determinant 1.

3. Given any three distinct points $P, Q, R \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation sending $P \to \infty, Q \to 0, R \to 1$, given by

$$\alpha(z) = \frac{(P-R)(Q-z)}{(Q-R)(P-z)}$$

(Uniqueness follows from the easy exercise that the only Möbius transformation fixing $0, 1$ and ∞ is the identity.) It follows that given any other three distinct points $P', Q', R' \in \hat{\mathbb{C}}$ there exists a unique Möbius transformation sending $P \to P'$, $Q \to Q'$ and $R \to R'$, for if α is as above and β sends $P' \to \infty, Q' \to 0, R' \to 1$ then $\beta^{-1}\alpha$ has the required property.

4. Given any four distinct points $P, Q, R, S \in \hat{\mathbb{C}}$, their *cross-ratio* is defined to be

$$(P, Q; R, S) = \frac{(P-R)(Q-S)}{(Q-R)(P-S)} \in \hat{\mathbb{C}} - \{0.1, \infty\}$$

(**Warning:** There are several different definitions of a cross-ratio in common use.) It follows from the preceding remark that $(P, Q; R, S) = \alpha(S)$, where α is the unique Möbius transformation sending $P \to \infty$, $Q \to 0$ and $R \to 1$. Hence if γ is a Möbius transformation then $(\gamma(P), \gamma(Q); \gamma(R), \gamma(S)) = (P, Q; R, S)$, for $\alpha\gamma^{-1}$ is then a Möbius transformation sending $\gamma(P) \to \infty$, $\gamma(Q) \to 0$, $\gamma(R) \to 1$ and has $(\alpha\gamma^{-1})\gamma(S) = \alpha(S) = (P, Q; R, S)$. Thus *cross-ratios are preserved by Möbius transformations.*

Möbius transformations are conformal (since they are invertible and therfore have non-zero derivative everywhere). But conformality is just a local property and we can prove a much stronger result:

Proposition 2.5 *Möbius transformations send circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$ (where a 'circle through ∞' is a straight line in $\mathbb{C}$).*

Proof Any 'circle' in $\hat{\mathbb{C}}$ (including those through ∞) has the form

$$\alpha(x^2 + y^2) + 2\beta x + 2\gamma y + \delta = 0 \quad (\alpha, \beta, \gamma, \delta \in \mathbb{R})$$

in other words

$$Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0$$

where $A = \alpha \in \mathbb{R}, B = \beta - i\gamma \in \mathbb{C}, C = \delta \in \mathbb{R}$. Let

$$z = \frac{aw + b}{cw + d}$$

Now a direct substitution for z in the equation above gives an equation of the same form for w once the denominator has been cleared. QED

Corollary 2.6 *Any four distinct points $P, Q, R, S \in \hat{\mathbb{C}}$ lie on a common circle if and only if their cross ratio $(P, Q; R, S)$ is real.*

Proof Send P, Q, R to $\infty, 0, 1$ by a Möbius transformation. QED

Proposition 2.7 *The conformal automorphisms of $\mathbb{C}$ are the maps $f(z) = az + b$ having $a, b \in \mathbb{C}$ and $a \neq 0$.*

Proof Let f be a conformal automorphism of $\mathbb{C}$. Then $\lim_{z\to\infty} f(z) = \infty$ (this follows from the fact that f is a homeomorphism). Hence $\sigma f \sigma$ has a removable singularity at 0 and so f extends to a conformal automorphism of $\hat{\mathbb{C}}$. The result follows by Proposition 2.4. QED

We next identify the conformal automorphisms of $\mathbb{D}$. The neatest method is via Schwarz's Lemma, which will be an important tool for us later for other purposes.

Lemma 2.8 (Schwarz's Lemma) *If f is holomorphic $\mathbb{D} \to \mathbb{D}$ and $f(0) = 0$ then $|f'(0)| \leq 1$. If $|f'(0)| = 1$ then $f(z) = \mu z$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. If $|f'(0)| < 1$ then $|f(z)| < |z|$ for all $0 \neq z \in \mathbb{D}$.*

Proof Let $f(z)$ have Taylor series $a_1 z + a_2 z^2 + ...$ on $\mathbb{D}$, and set $g(z) = a_1 + a_2 z + ...(= f(z)/z)$. Then g is holomorphic $\mathbb{D} \to \mathbb{C}$ and on the circle $\mathcal{C}$ having centre 0 and radius ρ we see that

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{\rho}$$

so by the Maximum Modulus Principle $|g(z)|$ has the bound $1/\rho$ for all z inside $\mathcal{C}$ too. Letting ρ tend to 1 (from below) we deduce that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, and in particular $|g(0)| \leq 1$, that is $|f'(0)| \leq 1$. If there is any $z_0 \in \mathbb{D}$ with $|g(z_0)| = 1$ (for example if $|g(0)| = 1$), then $|g(z)| = 1$ for all $z \in \mathbb{D}$ (again by the Maximum Modulus Principle) in which case g must be constant, say $g(z) = \mu$, with $|\mu| = 1$. If there is no such z_0 then $|g(z)| < 1$ for all $z \in \mathbb{D}$, i.e. $|f(z)| < |z|$ for all $z \in \mathbb{D}$. QED

Proposition 2.9 *The conformal automorphisms of $\mathbb{D}$ are the maps of form*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}, \quad \theta \in \mathbb{R}, a \in \mathbb{D}$$

Proof Let f be a conformal automorphism of $\mathbb{D}$. Then $f^{-1}(0) = a \in \mathbb{D}$. The Möbius transformation

$$g(z) = \frac{z - a}{1 - \bar{a}z}$$

sends a to 0 and the unit circle to itself, so it sends $\mathbb{D}$ to itself. Thus fg^{-1} is a conformal automorphism of $\mathbb{D}$ sending 0 to 0. From Schwarz's Lemma it follows that $fg^{-1}(z) = \mu z$ for some μ with $|\mu| = 1$. QED

Corollary 2.10 *The conformal automorphisms of the upper half-plane $\mathcal{H}_+$ are the Möbius transformations*

$$f(z) = \frac{az + b}{cz + d}$$

having $a, b, c, d \in \mathbb{R}$ and $ad \neq bc$.

Proof. Take any Möbius transform M which sends the upper half plane $\mathcal{H}_+$ bijectively onto $\mathbb{D}$ (exercise: write one down). The conformal automophisms of $\mathcal{H}_+$ are the maps $M^{-1}gM$ where g runs through the conformal automorphisms of $\mathbb{D}$ given by Proposition 2.9 (details: exercise). QED

We recall that not only is there is a conformal bijection between $\mathcal{H}_+$ and $\mathbb{D}$, but that the Riemann Mapping Theorem states that for every simply-connected domain $U \subset \mathbb{C}$ $(U \neq \mathbb{C})$ there is a conformal bijection between U and $\mathbb{D}$. An important generalisation of this that we shall repeatedly use explicitly or implicitly, but which we will not prove in this course, is the following (proved by Poincaré and Koebe):

Theorem 2.11 (The Uniformisation Theorem) *Every simply-connected Riemann surface is conformally bijective to one of $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$.*

2.5 The Poincaré metric on the upper half plane

Define the *infinitesimal Poincaré metric* on the upper half plane by $ds = \frac{\sqrt{(dx)^2+(dy)^2}}{y}$.

Proposition 2.12 *ds is invariant under $PSL(2, \mathbb{R})$*

Proof Every element of $PSL(2, \mathbb{R})$ can be written as a composition of transformations of the type $z \to z + \lambda$ $(\lambda \in \mathbb{R})$, $z \to \mu z$ $(\mu \in \mathbb{R}^{>0})$ and $z \to -1/z$, and it is easily checked that each preserves ds. QED

A path in $\mathcal{H}_+$ is called a *geodesic* from P to Q in $\mathcal{H}_+$ if it is a path of shortest length.

Proposition 2.13 *There is a unique geodesic between any two distinct points P and Q in $\mathcal{H}$. It is the segment between P and Q of the (unique) euclidean semicircle through P and Q which meets $\hat{\mathbb{R}}$ orthogonally. The distance between P and Q (in the Poincaré metric) is $ln(|(P, Q; A, B)|)$ where A and B are the points where the semicircle meets $\hat{\mathbb{R}}$.*

Proof In the case that P and Q are on the imaginary axis, the straight line path γ_1 from P to Q is shorter than any other path γ_2 from P to Q, since

$$\int_{\gamma_2} \frac{1}{y}\sqrt{(dx)^2 + (dy)^2} > \int_{\gamma_2} \frac{1}{y} dy = \int_{\gamma_2} \frac{1}{y} dy$$

For $P = i$ and $Q = it$ (real $t > 1$) the hyperbolic distance from P tp Q is

$$\int_1^t \frac{1}{y} dy = \ln t = \ln |(i, it; 0, \infty)|$$

The result follows, since given any P', Q' in $\mathcal{H}_+$ there is an element of $PSL(2, \mathbb{R})$ which sends P to P' and Q to Q', and moreover this Möbius transformation sends the positive imaginary axis to a semicircle with ends on the extended real axis $\hat{\mathbb{R}}$ and preserves corss-rations. QED

Corollary 2.14 *The group of conformal automorphisms of the upper half-plane, $PSL(2, \mathbb{R})$, is also the group of orientation-preserving isometries of the upper half-plane (equipped with the Poincaré metric).*

Proof (sketch) It is obvious that every element of $PSL(2, \mathbb{R})$ preserves the Poincaré metric since it preserves the upper half-plane, the real axis and cross-ratios. For the other direction, observe that an isometry of the Poincaré metric must send geodesics to geodesics, and it must send orthogonal pairs of geodesics to orthogonal pairs of geodesics (since othogonal pairs of geodesics are pairs of semicircles with end points on $\hat{\mathbb{R}}$ having cross-ratio -1). It follows that an isometry must satisfy the Cauchy-Riemann equations everywhere and is therefore a conformal automorphism. QED

We can transfer the Poincaré matric to $\mathbb{D}$, using any Möbius transformation M sending $\mathcal{H}_+ \to \mathbb{D}$.

Exercise. Show that the infinitesimal metric $\frac{2|dz|}{1-|z|^2}$ on $\mathbb{D}$ is invariant under $Aut(\mathbb{D})$, show that the distance between 0 and $t \in (0, 1) \subset \mathbb{D} \cap \mathbb{R}$ in this metric is $\ln |(0, t; -1, +1)|$ and deduce that this is the infinitesimal Poincaré metric, transferred from $\mathcal{H}_+$ to $\mathbb{D}$.

2.6 Conjugacies, fixed points and multipliers

Definition Rational maps f, g are said to be *conjugate* if there exists a Möbius transformation h such that $g = hfh^{-1}$, in other words such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ h\downarrow & & \downarrow h \\ \hat{\mathbb{C}} & \xrightarrow{g} & \hat{\mathbb{C}} \end{array}$$

Conjugate maps have identical dynamical behaviour (think of h as a 'change of coordinate system'). In particular h sends fixed points of f to fixed points of g, periodic points of f to periodic points of g etc, as we shall see below. We can often put a rational map into a simpler form by applying a suitable conjugacy.

Examples

1. A rational map f is conjugate to a polynomial if and only if there exists a point $z_0 \in \hat{\mathbb{C}}$ such that $f^{-1}(z_0) = \{z_0\}$. (**Proof**: Move z_0 to ∞ by a Möbius transformation h. Details: exercise.)

2. A rational map f is conjugate to a polynomial of the form $z \to z^n$ (some $n > 0$) if and only if there exist distinct points $z_0, z_1 \in \hat{\mathbb{C}}$ such that $f^{-1}(z_0) = \{z_0\}$ and $f^{-1}(z_1) = \{z_1\}$. (**Proof**: Move z_0 to ∞ and z_1 to 0 by a Möbius transformation h. Details: exercise.)

3. Every degree 2 polynomial $z \to \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$) is conjugate to a (unique) one of the form $z \to z^2 + c$. (**Proof**: Exercise: h can be taken of the form $az + b$ since we do not have to move ∞).

Fixed points and multipliers

Definitions A *fixed point* of a rational map f is a point $z_0 \in \hat{\mathbb{C}}$ such that $f(z_0) = z_0$.

The *multiplier* of f at such a fixed point is the derivative $f'(z_0) = \lambda$. We say that z_0 is

attracting if $|\lambda| < 1$ (if $\lambda = 0$ we say z_0 is *superattracting*);

repelling if $|\lambda| > 1$;

neutral if $|\lambda| = 1$, i.e. $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R}$.

As we shall see, the dynamical behaviour around a neutral periodic point depends on whether θ is rational or irrational, and the irrational case can be further subdivided into 'linearisable' and 'non-linearisable'.

Proposition 2.15 *When the function f is conjugated by a Möbius transformation h any fixed point z_0 of f is sent to a fixed point $w_0 = h(z_0)$ of $g = hfh^{-1}$ and the multiplier of the fixed point w_0 for g is equal to the multiplier for the fixed point z_0 for f.*

Proof If z_0 is a fixed point of f and $w_0 = h(z_0)$ then

$$g(w_0) = gh(z_0) = hf(z_0) = w_o$$

and, by the chain rule for differentiation,

$$g'(w_0) = h'(w_0)f'(z_0)(h^{-1})'(w_0)$$

but since h is differentiable, has differentiable inverse and sends z_0 to w_0, we know that

$$(h^{-1})'(w_0) = \frac{1}{h'(z_0)}$$

and hence $g'(w_0) = f'(z_0)$. QED

Note that we cannot expect the *derivative* of a rational map f at a point z_0 to be a conjugacy invariant when z_0 is not a fixed point, since there is no reason to expect any relation between $h'(z_0)$ and $(h^{-1})'(hf(z_0))$. However the property of having zero derivative does turn out to be a conjugacy invariant (exercise). This should not surprise us as this is a topological property of the map: the critical points are the *branch points* of the map, that is to say the points where it fails to be locally one-to-one.

Proposition 2.15 says that a conjugacy sends a fixed point of f to a fixed point of g having the same dynamical behaviour (attractor, repellor etc). Analogous results hold for periodic orbits:

Definition A point z_0 is said to *periodic* of period n for f if $f^n(z_0) = z_0$ but $f^j(z_0) \neq z_0$ for $0 < j < n$. The *multiplier* of the periodic orbit $\{z_0, f(z_0) = z_1, f(z_1) = z_2, ..., f(z_{n-1}) = z_0\}$ is defined to be $(f^n)'(z_0)$. Note that $(f^n)'(z_0) = f'(z_0)f'(z_1)...f'(z_{n-1})$ by the chain rule.

Proposition 2.16 *When the function f is conjugated by a Möbius transformation h any orbit of period n of f is sent an orbit of period n of $g = hfh^{-1}$, and the two orbits have the same multiplier.*

Proof Denote the periodic orbit of f by $\{z_0, f(z_0) = z_1, f(z_1) = z_2, ..., f(z_{n-1}) = z_0\}$. Then $g^j h(z_0) = hf^j(z_0) = h(z_j)$. So $g^j h(z_0) \neq h(z_0)$ for $0 < j < n$ (h being injective) and $g^n h(z_0) = h(z_0)$. Hence $h(z_0)$ is periodic of period n for g. The orbits have the same multiplier by Proposition 2.15 applied to f^n. QED

2.7 The spherical metric and the Fatou and Julia sets of a rational map

We define the *spherical metric* on the unit sphere S^2 by setting the distance between two points to be the shortest Euclidean length of a great circle path between them. On the Riemann sphere, parameterised as the extended complex plane $\mathbb{C} \cup \infty$, the infinitesimal spherical metric is:

$$ds = \frac{2|dz|}{1 + |z|^2}$$

WARNING The spherical metric is not preserved by $Aut(\hat{\mathbb{C}})$, but conjugating by any particular conformal automorphism sends the spherical metric to a Lipschitz equivalent metric, since $\hat{\mathbb{C}}$ is compact.

Definition Let f be a rational map and z_0 be a point of $\hat{\mathbb{C}}$. We say that the family of iterates $\{f^n\}_{n\geq 0}$ is *equicontinuous at z_0* if given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 0$ $d(f^n(z), f^n(z_0)) < \epsilon$ whenever $d(z, z_0) < \delta$. (Here d is the spherical metric on $\hat{\mathbb{C}}$).

Think of this as saying that *'any orbit which that starts near z_0 remains close to the orbit of z_0'*.

Definitions The *Fatou set* $F(f)$ of f is the largest open subset of $\hat{\mathbb{C}}$ on which the family $\{f^n\}_{n\geq 0}$ is equicontinuous at every point. The *Julia set* $J(f)$ of f is $\hat{\mathbb{C}} - F(f)$.

The Julia set should be thought of as the set of points the orbits of which exhibit *'sensitive dependence on initial conditions'*.

Example

$f(z) = z^2$ has Fatou set $F(f) = \{z : |z| \neq 1\}$, and Julia set $J(f) = \{z : |z| = 1\}$.

Since f doubles length along the unit circle it is clear that $\{z : |z| = 1\} \subset J(f)$. It is not quite so obvious that points not on the unit circle are in $F(f)$. One can give a direct formal proof of this, but the details are a little messy in practice: the problem is that orbits started close together near (but not on) the unit circle will move apart for a large number of iterations before they start approaching each other again. For a more general method of proof, see the next chapter.

Remark If $g = hfh^{-1}$ where $h \in Aut(\hat{\mathbb{C}})$, then $F(g) = h(F(f))$ and $J(g) = h(J(f))$. This follows from the remark about Lipschitz equivalent metrics in the warning above.