Recurrence and transience of some contractive Markov chains with super-heavy tailed innovations in random environment

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Recurrence/transience criteria for

(1) 1-dimensional

(a) Non-negative contractive autoregressive processes of order 1:

$$X_n := aX_{n-1} + Y_n$$

$0 < a < 1$, $(Y_n)_{n\geq 1}$ i.i.d., $[0, \infty)$-valued

(b) Subcritical Galton-Watson processes with immigration

(2) Generalization of (1) to $d \geq 1$-dimensional processes

(3) Generalization of (2) to random environments

Theorem 1: (Random exchange process[1], long range percolation; Lamperti, Kesten '70; Kellerer '92, '06) *Let W_n, $n \geq 1$, be i.i.d. $\mathbb{N}_0$-valued, $P[W_1 = 0] > 0$.*

$$\textit{Set} \quad R_0 := 0 \quad \textit{and} \quad R_n := \max\{R_{n-1} - 1, W_n\} \quad \textit{for } n \geq 1.$$

Lamperti:

$$(R_n)_{n\geq 0} \textit{ is transient if } \liminf_{n\to\infty} n\, P[W_1 > n] > 1 \quad \textit{and}$$

$$(R_n)_{n\geq 0} \textit{ is recurrent if } \limsup_{n\to\infty} n\, P[W_1 > n] < 1.$$

Kesten; Kellerer:

$$(R_n)_{n\geq 0} \textit{ is recurrent iff } \sum_{n\geq 0} \prod_{m=0}^{n} P[W_1 \leq m] = \infty. \qquad (*)$$

Proof of $(*)$: For all $n \geq 1$ it holds that $R_n = \max_{m=1}^{n}(W_m - n + m) \geq 0$. The state 0 (and by irreducibility the Markov chain) is recurrent iff $G(0,0) = \infty$, i.e. iff

$$\infty = \sum_{n\geq 1} P[R_n = 0] = \sum_{n\geq 1} P\left[\max_{m=1}^{n}(W_m - n + m) \leq 0\right] \overset{\text{indep.}}{=} \sum_{n\geq 1} \prod_{m=1}^{n} P[W_m \leq n - m]$$

$$\overset{\text{i.d.}}{=} \sum_{n\geq 1} \prod_{m=1}^{n} P[W_1 \leq n - m] = \sum_{n\geq 1} \prod_{m=0}^{n-1} P[W_1 \leq m] = \sum_{n\geq 0} \prod_{m=0}^{n} P[W_1 \leq m]. \qquad \square$$

[1] Gade. Deep water exchanges in a sill fjord: a stochastic process. *J. Phys. Oceanography*, 1973

$$\left.\begin{array}{l}(W_n)_n \text{ iid}, W_1 \in \mathbb{N}_0 \text{ a.s.}, P[W_1=0]>0\\ R_n = \max\{R_{n-1}-1, W_n\}\end{array}\right\} (R_n)_n \text{ rec.} \Leftrightarrow \sum_{n\geq 0}\prod_{m=0}^{n} P[W_1 \leq m] = \infty$$

$$\left.\begin{array}{l}(W_n)_n \text{ iid}, W_1 \geq 0 \text{ a.s.}, P[W_1=0]>0\\ R_n = \max\{R_{n-1}-1, W_n\}\end{array}\right\} (R_n)_n \text{ rec.} \Leftrightarrow \sum_{n\geq 0}\prod_{m=0}^{n} P[W_1 \leq m] = \infty$$

$$\left.\begin{array}{l}(W_n)_n \text{ iid}, W_1 \geq 0 \text{ a.s.}, P[W_1=0]>0\\ R_n = \max\{R_{n-1}-c, W_n\}, c>0\end{array}\right\} (R_n)_n \text{ rec.} \Leftrightarrow \sum_{n\geq 0}\prod_{m=0}^{n} P[W_1 \leq mc] = \infty$$

$$\left.\begin{array}{l}(W_n)_n \text{ iid}, W_1 \geq 0 \text{ a.s.}, P[W_1\leq w]>0\\ R_n = \max\{R_{n-1}-c, W_n\}, c>0, w\geq 0\end{array}\right\} (R_n)_n \text{ rec.} \Leftrightarrow \sum_{n\geq 0}\prod_{m=0}^{n} P[W_1 \leq w+mc] = \infty$$

$$\Updownarrow$$

Max-autoregressive process $(M_n)_n := (e^{R_n})_n$ rec. $\quad (a := e^{-c}, Y_n := e^{W_n}, y := e^{w})$

$$\left.\begin{array}{l}(Y_n)_n \text{ iid}, Y_1 \geq 1 \text{ a.s.}, P[Y_1\leq y]>0\\ M_n = \max\{aM_{n-1}, Y_n\}, 0<a<1\end{array}\right\} (M_n)_n \text{ rec.} \Leftrightarrow \sum_{n\geq 0}\prod_{m=0}^{n} P[Y_1 \leq ya^{-m}] = \infty$$

$$\Updownarrow ?$$

$$\left.\begin{array}{l}(Y_n)_n \text{ iid}, Y_1 \geq 0 \text{ a.s.}, P[Y_1\leq y]>0\\ X_n = aX_{n-1}+Y_n, 0<a<1\end{array}\right\} (X_n)_n \text{ rec.}$$

Definition of recurrence/transience (see e.g. Kellerer 2006 for a more general setting):

Let $\mathcal{H}$ be the set of continuous functions from $[0,\infty)^d$ to $[0,\infty)^d$ which are monotone with respect to the partial order $\leq$ on $[0,\infty)^d$.

- A $[0,\infty)^d$-valued Markov chain $V=(V_n)_{n\geq 0}$ with initial state $V_0=0$ (unimportant) is order-preserving iff it fulfills a recursion of the form $V_n=H_n(V_{n-1})$ for an i.i.d. sequence $(H_n)_{n\geq 1}$ of $\mathcal{H}$-valued random variables.
- If V is order-preserving then V is called irreducible iff for all $x\in[0,\infty)^d$ there is some $n\geq 0$ such that $P[V_n\geq x]>0$.
- An irreducible V is called recurrent iff there exists $b\in(0,\infty)$ such that $\sum_{n\geq 0} P[\|V_n\|\leq b]=\infty$. Otherwise V is called transient.

Fact: Let V be irreducible.
Then V is recurrent iff there is a $b<\infty$ such that a.s. $\|V_n\|\leq b$ infinitely often.
V is transient iff a.s. all components of V diverge to ∞.

Theorem 2: (Z., 2016) *Let* $0 < a < 1$, $y \geq 0$ *and let* $(Y_n)_{n\geq 1}$ *be an i.i.d. sequence of non-negative random variables such that* $P[Y_1 \leq y] > 0$. *Set* $X_0 := 0$ *and*

$$X_n := aX_{n-1} + Y_n$$

for all $n \geq 1$. *Then* $(X_n)_{n\geq 0}$ *is recurrent iff*

$$\sum_{n\geq 0} \prod_{m=0}^{n} P[Y_1 \leq ya^{-m}] = \infty. \qquad (*)$$

Earlier results:

Theorem 3: (Kellerer 1992, unpublished) $(X_n)_{n\geq 0}$ *is*

$$\begin{aligned} &\textit{transient if} \quad \liminf_{t\to\infty} t \cdot P[\ln Y_1 > t] \;>\; -\ln a \quad \textit{and} \\ &\textit{recurrent if} \quad \limsup_{t\to\infty} t \cdot P[\ln Y_1 > t] \;<\; -\ln a. \end{aligned}$$

Theorem 4: (Zeevi, Glynn 2004) *Let* $P[\ln(1+Y_1) > t] = (1+\beta t)^{-p}$ *for some* $\beta > 0$ *and* $p > 0$ *("log-Pareto"). Then* $(X_n)_{n\geq 0}$ *is recurrent if* $p > 1$ *or* ($p = 1$ *and* $\beta \ln(1/a) \geq 1$), *and transient otherwise.*

Theorems 3 and 4 follow from Theorem 2 and Raabe's test.

Pf Th 1: Claim: $X_n = aX_{n-1} + Y_n$ recurrent $\Leftrightarrow$ $M_n = \max\{aM_{n-1}, Y_n\}$ recurrent $\Leftrightarrow$

$$\sum_{n\geq 0} \prod_{m=0}^{n} P[Y_1 \leq ya^{-m}] = \infty \qquad (*)$$

M recurrent $\Leftrightarrow (*)$: ✓

X recurrent $\Leftrightarrow$ M recurrent: „$\Rightarrow$": $X_n \geq M_n$.

„$\Leftarrow$": Let $b < \infty$ be such that $\sum_{n\geq 0} P[M_n \leq b] = \infty$ and $P[Y_1 \leq b] \geq 1/2$. Set

$\tau := \inf\{m \geq 0 : a^m Y_1 \leq b\}$. Since $X_n = \sum_{m=1}^{n} a^{n-m} Y_m$ and $M_n = \max_{m=1}^{n} a^{n-m} Y_m$,

$$\begin{aligned} E[X_n \,|\, M_n \leq b] &= \sum_{m=1}^{n} E\left[a^{n-m} Y_m \,\middle|\, \bigcap_{i=1}^{n} \{a^{n-i} Y_i \leq b\}\right] = \sum_{m=1}^{n} E\left[a^{n-m} Y_m \,\middle|\, a^{n-m} Y_m \leq b\right] \\ &= \sum_{m=0}^{n-1} \frac{E\left[a^m Y_1, a^m Y_1 \leq b\right]}{P[a^m Y_1 \leq b]} \leq 2E\left[Y_1 \sum_{m\geq \tau} a^m\right] = \frac{2E[a^\tau Y_1]}{1-a} \leq \frac{2b}{1-a} =: c. \end{aligned}$$

Therefore,
$$\begin{aligned} P[X_n \leq 2c] &\geq P[X_n \leq 2c, M_n \leq b] = P[M_n \leq b] - P[X_n > 2c, M_n \leq b] \\ &\geq P[M_n \leq b] - \frac{E[X_n, M_n \leq b]}{2c} \geq \frac{P[M_n \leq b]}{2}, \end{aligned}$$

which is not summable. □

Galton-Watson process $(Z_n)_{n\geq 0}$ with immigration $(Y_n)_{n\geq 0}$: Heathcote 1965.

Let $(X_{n,j})_{n\geq 0, j\geq 1}$ be i.i.d. and independent of the i.i.d. sequence $(Y_n)_{n\geq 1}$ with $P[X_{1,1}\in\mathbb{N}_0] = 1 = P[Y_1\in\mathbb{N}_0]$. Set $Z_0 := 0$ and

$$Z_n := Y_n + \sum_{j=1}^{Z_{n-1}} X_{n-1,j} \quad \text{for all } n\geq 1.$$

Pakes 1975, 1979: Sufficient conditions for recurrence or transience of Z in terms of generating functions

Theorem 5: (Z. 2016) *Assume* $0 < a := E[X_{1,1}] < 1$ *and* $E[X_{1,1}\log X_{1,1}] < \infty$ *and let* $y\in(0,\infty)$ *be such that* $P[Y_1\leq y] > 0$. *Then* $(Z_n)_{n\geq 0}$ *is recurrent iff*

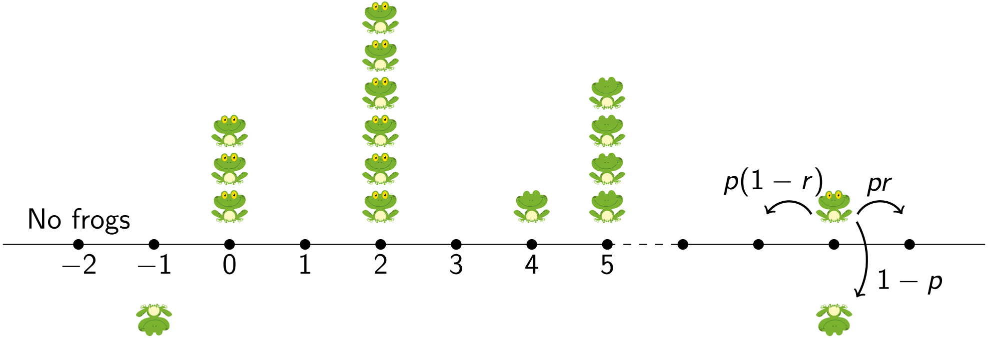
$$\sum_{n\geq 0}\prod_{m=0}^{n} P[Y_1\leq ya^{-m}] = \infty.$$

The proof uses that

- $X_n := E[Z_n \,|\, (Y_m)_{m\geq 1}]$ defines an autoregressive process: $X_n = Y_n + aX_{n-1}$
- $P[\text{GW-process with average offspring } a \text{ is not extinct at time } n] \sim \textit{const } a^n$.

Application of Theorem 5 to frog processes

Fix $p, r \in (0,1)$. Put on each $n \geq 0$ a number Y_n of sleeping frogs. Wake up the frogs at 0. Once woken up, every frog performs a nearest-neighbor random walk, jumping independently of everything else with probability pr to the right and with probability $p(1-r)$ to the left. In each step it dies with probability $1-p$. Whenever a frog visits a site with sleeping frogs those frogs are woken up and start their own independent lives.



Corollary 6: (Z. 2016) *Let $y \in (0, \infty)$ be such that $P[Y_0 \leq y] > 0$. Then the following statements are equivalent.*

- *Almost surely only finitely many different frogs visit 0.*
- *Almost surely only finitely many frogs are woken up.*
- $\sum_{n\geq 0} \prod_{m=0}^{n} P\left[Y_0 \leq ya^{-m}\right] = \infty$, *where* $a := \dfrac{1-\sqrt{1-4p^2r(1-r)}}{2p(1-r)}$ *is the probability that a frog starting at 0 ever reaches 1.*

Proof: $Z_n :=$ # frogs ever jumping from n to $n+1$ gives a subcritical branching process with immigration up to the first time of extinction

(2) General dimension $d \geq 2$:

Theorem 7: (Z. 2016) *Let* $A \in [0, \infty)^{d \times d}$ *be primitive with spectral radius* $\rho < 1$*,* $y \in (0, \infty)$ *and let* $(Y_n)_{n \geq 1}$ *be an i.i.d. sequence of* $[0, \infty)^d$*-valued random variables such that* $P[\|Y_1\| \leq y] > 0$*. Set* $X_0 := 0$ *and*

$$X_n := AX_{n-1} + Y_n$$

for all $n \geq 1$*. Then* $(X_n)_{n \geq 0}$ *is recurrent iff*

$$\sum_{n \geq 0} \prod_{m=0}^{n} P[\|Y_1\| \leq y\rho^{-m}] = \infty.$$

A similar statement holds for multitype Galton-Watson processes with immigration.

(3) Random environment:

Theorem 8: (Z. 2016) *Let $(Y_n)_{n\geq 0}$ be a sequence of $[0,\infty)^d$-valued random vectors and let $(A_n)_{n\geq 1}$ be a sequence of $[0,\infty)^{d\times d}$-valued random matrices. Assume that $(A_n, Y_n)_{n\geq 1}$ is i.i.d.. Let $y \in (0,\infty)$ be such that $P[\|Y_1\| \leq y] > 0$. Define the a.s. limit*

$$\lambda := \lim_{n\to\infty} \frac{-\ln \|A_1 \dots A_n\|}{n} \quad (= -E[\ln A_1] \text{ if } d = 1)$$

and assume $\lambda > 0$. Assume a certain boundedness condition (BA) on A_1 and a regularity condition (REG) on Y_1. Set $X_0 := 0$ and

$$X_n := A_n X_{n-1} + Y_n \qquad \text{for all } n \geq 1.$$

Then $(X_n)_{n\geq 0}$ is recurrent iff $$\sum_{n\geq 0} \prod_{m=0}^{n} P[\|Y_1\| \leq ye^{m\lambda}] = \infty.$$

A similar statement holds for multitype Galton-Watson processes $(Z_n)_{n\geq 0}$ with immigration in random environment. Earlier results:

Theorem 9 (Bauernschubert 2013): *If $d = 1$ then under weak assumptions*

$(Z_n)_{n\geq 0}$ *(and $(X_n)_{n\geq 0}$) is transient if* $\liminf_{t\to\infty} t \cdot P[\ln Y_1 > t] > -E[\ln A_1]$ *and*

$(Z_n)_{n\geq 0}$ *is recurrent if* $\limsup_{t\to\infty} t \cdot P[\ln Y_1 > t] < -E[\ln A_1]$.

Assumptions in Theorem 8:

$$\begin{array}{l}\text{If } d=1 \text{ then } \ln A_1 - E[\ln A_1] \text{ is sub-Gaussian.}\\ \text{If } d\geq 2 \text{ then there exist } K,\gamma\in\mathbb{N} \text{ and } \kappa>0 \text{ such that a.s.}\\ \|A_1\|\leq\gamma \text{ and } A_1\dots A_K\in[\kappa,\infty)^{d\times d}.\end{array} \tag{BA}$$

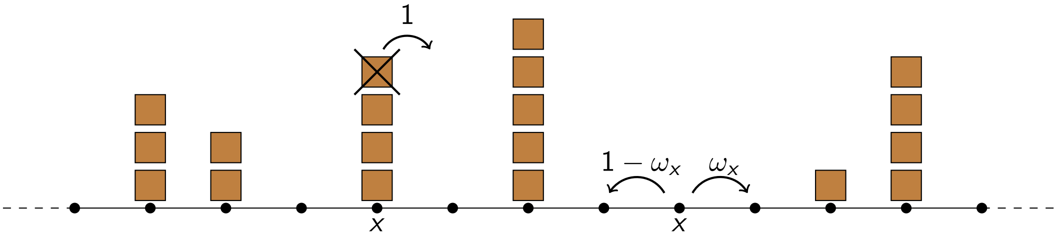
$$\lim_{x\to\infty} x^{2/3}(\ln x)^2 P[\|Y_1\|>e^x]=0 \qquad \text{or} \qquad \liminf_{x\to\infty} xP[\|Y_1\|>e^x]>\lambda. \tag{REG}$$

Main tool for the proof of Theorem 8:

Lemma 10 (Sub-Gaussian concentration inequality) *Assume* (BA) *and set* $S_n := -\ln\|A_1\dots A_n\|$. *Then there are constants* c_1 *and* c_2 *such that for all* $n\in\mathbb{N}$ *and* $t\in(0,\infty)$,

$$P\left[|S_n-\lambda n|\geq t\right]\leq c_1\exp\left(-c_2t^2/n\right).$$

Application of Theorem 8 to excited random walks in random environment:
Let $\omega_x, x \in \mathbb{Z}$, be i.i.d. (0,1)-valued. If there are no cookies at the walker's current position x then the walker jumps independently of everything else with probability ω_x to $x+1$ and with probability $1-\omega_x$ to $x-1$. This walk (without any cookies at all) is a.s. transient to the left iff $E[\ln(1-\omega_0)/\omega_0] > 0$ (Solomon '75, Smith-Wilkinson '69).



Corollary 11: (Bauernschubert 2013, Z. 2016) *Assume that* $(\omega_x, Y_x)_{x\in\mathbb{Z}}$ *is i.i.d.,* $E[\ln(1-\omega_0)/\omega_0] > 0$, $\ln(1-\omega_0)/\omega_0 - E[\ln(1-\omega_0)/\omega_0]$ *sub-Gaussian,* (REG), *and* $P[Y_0 = 0] > 0$.

(a) *If* $E[\ln_+ Y_0] < \infty$ *then the walk is a.s. transient to the left.*

(b) *If* $E[\ln_+ Y_0] = \infty$ *and if*

$$\sum_{n\geq 0}\prod_{m=0}^{n} P\left[Y_0 \leq \exp\left(mE\left[\ln\frac{1-\omega_0}{\omega_0}\right]\right)\right] = \infty$$

then the walk is a.s. recurrent.

(c) *If the series in* (b) *is finite then the walk is a.s. transient to the right.*

Proof: $Z'_n := \#$ upcrossing from n to $n+1$. Then $Z'_n = Z_n$ if the first excursion to the right is finite and $Z'_n \leq Z_n$ otherwise.