Noise stability of matrix spectra

Ofer Zeitouni

London
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A surprising empirical fact

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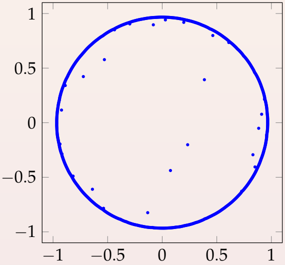

Background: Spectrum stability for symmetric matrices

A – matrix with singular values denoted $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$,

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In particular, if W is a symmetric matrix with i.i.d. centered standard Gaussian entries on and above diagonal (**Wigner matrix**), then $\lambda_{\max}(N^{-1/2}W) \to 2$, and if $\gamma > 1/2$ then

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No such control holds for eigenvalues of non-Hermitian matrices.

Background II: Ginibre matrices

Denote by G_N matrix with i.i.d. standard complex Gaussian entries, and set $g_N = N^{-1/2} G_N$.
λ_i - eigenvalues of g_N.
$L_N^g = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ - *empirical measure* of eigenvalues.

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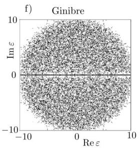
Theorem

L_N^g converges to the uniform measure on the unit disc.



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Thus, $\|N^{-\gamma} G_N\| \to 0$ if $\gamma > 1/2$.

Regularization by noise

Consider the nilpotent N-by-N matrix

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Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues L_N^A. Then L_N^A converges weakly to the uniform measure on the unit circle in the complex plane.

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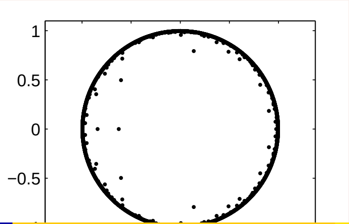

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Why is this particular perturbation picked up?

Noise Stability-Nilpotent matrices

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If $b = a \log N$ and γ is large enough, then the spectral radius of $J_{b,N} + N^{-\gamma} G_N$ is uniformly strictly smaller than 1. In particular,

$$L_N^{J_{a \log N,N} + N^{-\gamma} G_N} \not\to \delta_{S^1}$$

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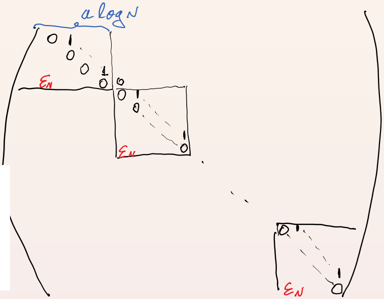

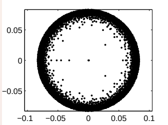

$\varepsilon_N = N^{-\gamma}$

In block: $\left(N^{-\gamma}\right)^{\frac{1}{a \log N}} \simeq e^{-\gamma/a}$

Q $\delta \lesssim e^{-\gamma/a}$?

Noise Stability-Block Nilpotent III

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Noise Stability-Block Nilpotent IV

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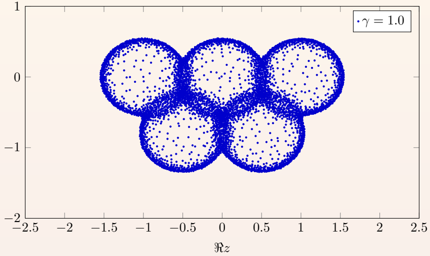

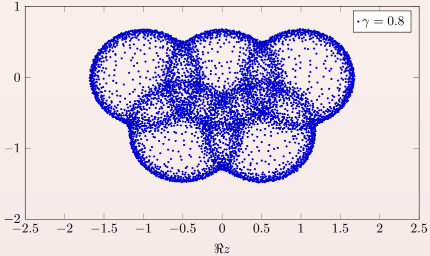

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Analogous result for $\gamma \in (1/2, 1]$ if collection of circles "does not spread too much" (e.g., olympics rings example OK).

More general models?

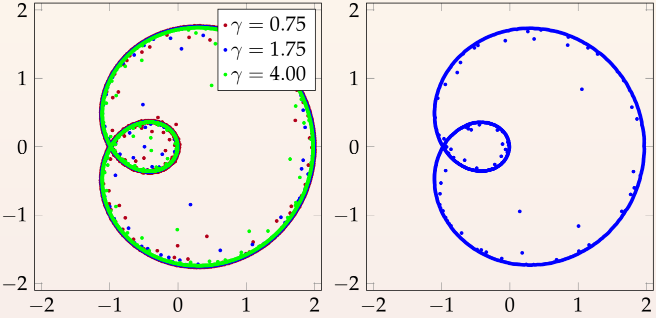

Figure: The eigenvalues of $J_N + J_N^2 + N^{-\gamma} G_N$, with $N = 4000$ and various γ. On left, actual matrix. On the right, $U_N(J_N + J_N^2)U_N^*$.

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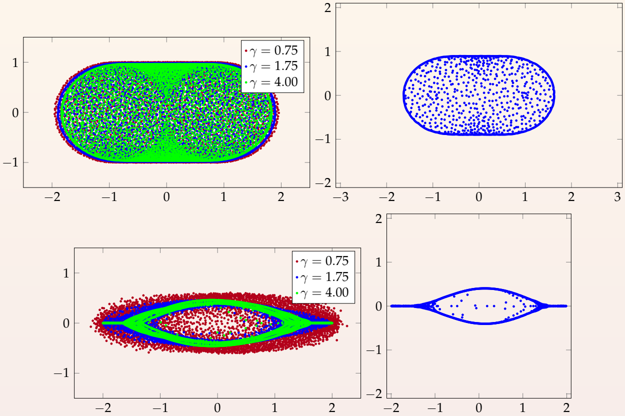

Figure: The eigenvalues of $D_N + J_N + N^{-\gamma} G_N$, with $N = 4000$ and various γ. Top: $D_N(i,i) = -1 + 2i/N$. Bottom: D_N i.i.d. uniform on $[-2, 2]$. On left, actual matrix. On the right, $U_N(D_N + J_N)U_N^*$.

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b) $T_N = \sum_{i=0}^k a_i J_N^i$ *(Toeplitz, finite symbol). Then,*

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Some two-diagonal Toeplitz cases studied by Sjöstrand and Vogel (2016)

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For matrix A_N, empirical measure of eigenvalues: L_N^A.
Let $L_N^{z,A}$ be empirical measure of eigenvalues of $B_N = (zI - A_N)(zI - A_N)^*$.

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Elements in proofs

Basic idea (Girko, Bai, Gotze-Tikhomirov, Tao-Vu, ...)
Logarithmic potential: $U_\mu(z) = \int \log|z-x|\mu(dx)$.
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But B_N is Hermitian, and so easier to work with!

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A_N - $N \times N$ matrix, uniformly bounded in operator norm.

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Definitions: A_N converges in $*$-moments toward an element a in a W^* probability space $(\mathcal{A}, \|\cdot\|, *, \phi)$ (faithful trace ϕ) if for any non-commutative polynomial P,

$$\frac{1}{N}\mathrm{tr}P(A_N, A_N^*) \xrightarrow{N\to\infty} \phi(P(a, a^*))\,.$$

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In particular $\log\det(z - a) = \int \log x \; d\nu_a^z(x) \quad z \in \mathbb{C}$, where ν_a^z denotes the spectral measure of the operator $|z - a|$.

Śniady's theorem

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Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2} G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2} G_N$;

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How can we take $t = t_N \to 0$?

Noise Stability-Maximal Nilpotent

$a \in \mathcal{A}$ is regular if for f smooth, compactly supported,

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The proof uses the regularity (of the limit) to truncate the singularity of the log... and depends crucially on convergence to ν_a. But it is not useful in maximally nilpotent example, since $L_N^A = \delta_0 \not\to \nu_a = \delta_{S^1}$!

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So it is enough to find a perturbation with correct limiting behavior! Nilpotent example uses a- unitary element (which is regular), E_N is $(N, 1)$ element.

Noise Stability-Block Nilpotent IV

A_N block matrix, each block of size $a_i \log N$.

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Analogous result for $\gamma \in (1/2,]$ if collection of circles "does not spread too much" (e.g., olympics rings example OK).

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Expand determinant and identify dominant terms using concentration of measure.

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Write $zI - M_N = U\Sigma_N V^*$, Σ_N - diagonal, singular values, arranged non-decreasing, and then

$$\Sigma = \Sigma_N = \begin{pmatrix} S_N & \\ & B_N \end{pmatrix}, \quad N^{-\gamma} G_N = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

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So only need to understand small singular values of M_N.

The Toeplitz case - transfer matrices

Reformulation: $M_N = \sum_{i=1}^{k} a_i J_N^i$, singular values?

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Reformulation: $M_N = \sum_{i=1}^{k} a_i J_N^i$, singular values? (allow a_i to change by row.) Set $V = \{x \in R^N : ((M_N - zI_N)x)_j = 0, j = 1, \ldots, N-k\}$. Parametrized by $x_1, \ldots, x_k$ and transfer matrices $T_j(z)$

$$(x_\ell)_{\ell=j+1}^{j+k} = T_j(z)(x_\ell)_{\ell=j}^{j+k-1}.$$

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$\mu_i(z)$ - Lyapunov exponents for $\prod_i T_i(z)$. If $T_i(z) = T(z)$ (Toeplitz) - modulii of eigenvalues of $T(z) = T_j(z)$, which are the roots of the symbol $P(z) = \sum_{i=0}^k a_i z^i$. The BPZ theorem can then be reformulated.

Theorem

$$U_{L_N}(z) \to \frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{i\theta}) - z| dz$$

and therefore

$$U_{L_N}(z) \to \int_0^{2\pi} \log|P(e^{i\theta}) - z| dz = \log|a_k| + \sum_{i=1}^k [(\log|\mu_i(z)|) \vee 0].$$

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Lemma

Take $d_i = D_{ii}$ iid.
a) If $E \log |z - d_1| < 0$ then

$$N^{-1} \log \sigma_N(M_N) \to E \log |z - d_1|,$$

$$\sigma_{N-1}(M_N) \geq N^{-C}.$$

b) If $E \log |z - d_1| > 0$ then $\sigma_N(M_N) \geq N^{-C}$.

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In particular $N^* = 1$.
Since

$$N^{-1} \sum_{i=1}^{N} \log \sigma_i(M_N) = N^{-1} \log \det(M_N),$$

get that log-potential converges to $(E \log |z - d_1|) \vee 0$.

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Suppose $D_N = 0$, $d_i = -z$ is constant.

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If $|z| \neq 1$, $\sigma_{N-1}(M_N) \geq C(z)(\log N)^{-1}$.
If $|z| < 1$ *then* $C_2 < \sigma_N(M_N)|z|^{-N} < C_1$.

BPZ - two diagonal case - $M_N = -zI + D_N + J$

Suppose $D_N = 0$, $d_i = -z$ is constant.

Lemma

If $|z| \neq 1$, $\sigma_{N-1}(M_N) \geq C(z)(\log N)^{-1}$.
If $|z| < 1$ *then* $C_2 < \sigma_N(M_N)|z|^{-N} < C_1$.
If $|z| > 1$ *then* $\sigma_N(M_N) > C/\log N$.

This reconstructs the GWZ result!

BPZ - two diagonal case - $M_N = -zI + J_N$

Assume $|z| < 1$. Set $v_1 = 1$, $v_k = zv_{k-1}$. Then

$$(M_N v)_k = \begin{cases} 0, & k \le N-1, \\ -z^N, & k = N. \end{cases}$$

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Let $\|x\|_2 = 1$, for $k \le N-1$, $x_{k+1} = zx_k + (M^N x)_k$, hence for $a \in \mathbb{C}$,

$$x_k - av_k = (x_1 - av_1)z^{k-1} + \sum_{j=1}^{k-1} (M^N x)_j z^{k-j}$$

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But if $\|x\|_2 = 1$ and $\langle x, v\rangle = 0$, get $1 \le \|x - av\|$. Choose $x_1 - av_1 = 0$, get a lower bound on $\|\pi M_N x\|$ in terms of $\|x - av\|$ of the form

$$\|M_N x\| \ge C(z)/\log N \Rightarrow \sigma_{N-1}(M_N) \ge C(z)/\log N.$$

BPZ - two diagonal case - $M_N = -zI + D_N + J_N$

To control $\sigma_N(M_N)$, recall

$$\sigma_N(M_N) = \inf_x \frac{\|M_N x\|_2}{\|x\|_2}$$

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Write $x = av/\|v\|_2 + by$ with $\langle y, v\rangle = 0$, $\|y\|_2 = 1$. Then with π the projection to first $N-1$ coordinates,

$$\|M_N x\|_2^2 = \|\pi M_N y\|_2^2 |b|^2 + ((M_N x)_N)^2.$$

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Since $a^2 + b^2 = 1$, either b is large or $M_N v$ is small. Algebraic manipulations give

$$\sigma_N(M_N) \geq C \inf_{x:\langle x,v\rangle=0} \frac{\|\pi M_N x\|_2}{\|x\|_2} \cdot \frac{\|M_N v\|_2}{\|v\|_2} \geq \frac{c|z|^N}{\log N}$$

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Slowly varying diagonals: apply this argument in blocks.
Multi diagonal: choose appropriate basis according to composition of eigenvalues of transfer matrix.