Local eigenvalue statistics of 1d random band matrices

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CLASSICAL AND QUANTUM MOTION IN DISORDERED ENVIRONMENT
A random event in honour of Ilya Goldsheid's 70-th birthday
Queen Mary, University of London, 19/12/2017

Band matrices: simplest model

H - hermitian or real symmetric $N \times N$ matrices with independent (up to the symmetry condition) entries H_{ij} such that

$$E\{H_{ij}\} = 0, \quad Var\{H_{ij}\} = (2W)^{-1} 1_{|i-j|\le W}$$

$$H = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

We are going to study the regimes

$$W \to \infty, \quad W/N \to 0, \quad \text{as} \quad N \to \infty,$$

Band matrices: general definition

H - hermitian or real symmetric $N \times N$ matrices with independent (up to the symmetry condition) entries H_{ij} such that $E\{H_{ij}\} = 0$,

$$E\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}W^{-d}J((i-j)/W), \quad i,j \in \mathbb{Z}^d$$

and $J \in L_1(\mathbb{R}^d)$ is a piece-wise continuous function (with a finite number of jumps), satisfying the conditions

$$J(x) = J(|x|), \quad 0 \le J(x) \le C, \quad W^{-d}\sum_j J(j/W) \to 1, \text{u is continuous at } x = 0$$

Our model-1 (RBM)

$$\mathbb{E}\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}\big(-W^2\Delta + 1\big)^{-1}_{ij} \sim W^{-1}e^{-|i-j|/W},$$

Our model-2: 1d Wegner type band matrix (RBBM)

H is $N \times N$ hermitian block matrix composed from n^2 blocks of the size $W \times W$ $(N = nW)$. Only 3 block diagonals are non zero.

$$H = \begin{pmatrix} A_1 & B_1 & 0 & 0 & 0 & \dots & 0 \\ B_1^* & A_2 & B_2 & 0 & 0 & \dots & 0 \\ 0 & B_2^* & A_3 & B_3 & 0 & \dots & 0 \\ . & . & B_3^* & . & . & . & . \\ . & . & . & . & . & A_{n-1} & B_{n-1} \\ 0 & . & . & . & 0 & B_{n-1}^* & A_n \end{pmatrix}$$

where

$A_1, \dots A_n$ - independent $W \times W$ GUE-matrices with entry's variance $(1 - 2\alpha)/W, \quad \alpha < \frac{1}{4}$

$B_1, \dots B_{n-1}$ -independent $W \times W$ Ginibre matrices with entry's variance α/W

Global regime: results

Let $\{\lambda_i\}_{i=1}^N$ be eigenvalues of H. Define linear eigenvalue statistics of the test function h as

$$\mathcal{N}_N[h] = \sum h(\lambda_i)$$

Limit of NCM ([Molchanov,Khorunzhy,Pastur:92])

$$\lim_{N,W\to\infty} N^{-1}\mathcal{N}_N(h) = \int h(\lambda)\rho(\lambda)d\lambda,$$

$$\text{where} \quad \rho(\lambda) = 1_{[-2,2]}(2\pi)^{-1}\sqrt{4-\lambda^2}$$

Theorem [MS:15]

If $h \in \mathcal{H}_s$ with $s > 2$, then

$$\sqrt{W/N}(\mathcal{N}_N[h] - E\{\mathcal{N}_N[h]\}) \to V(u)\mathcal{N}(0,1)$$

Previous results:
L.Li, A. Soshnikov (2013), and I. Jana, K. Saha, and A.Soshnikov (2014): CLT for band matrices with $W^2 >> N$;

"Anderson transition" for random band matrices (conjectures)

Let ℓ be a typical localization length of eigenvectors of H.

Localization and delocalization regimes

Localization regime means that $\ell << N$ and delocalization regime means that $\ell \sim N$. Varying W, we can see the crossover between localization and delocalization regimes.

$W = O(1)$ [$\sim$ random Schrödinger] $\longleftrightarrow$ $W = N$ [Wigner matrices]

Conjecture (in the bulk of the spectrum):

$d = 1:$	$\ell \sim W^2$	$W^2 \gg N$	Delocalization, local GUE statistics
		$W^2 \ll N$	Localization, Poisson statistics
$d = 2:$	$\ell \sim e^{W^2}$	$W^2 \gg \log N$	Delocalization, local GUE statistics
		$W^2 \ll \log N$	Localization, local Poisson statistics
$d \geq 3:$	$\ell \sim N$	$W \geq W_0$	Delocalization, local GUE statistics

Second order correlation function

$$R_2(\lambda_1, \lambda_2) = \int p_N(\lambda_1, \ldots, \lambda_N) d\lambda_3 \ldots d\lambda_N,$$

where $p_N(\lambda_1, \ldots, \lambda_N)$ is a joint eigenvalue distribution.

$$R_2(\lambda_1, \lambda_2) = \lim_{\varepsilon \to 0} (\pi N)^{-2} E\{\Im \mathrm{Tr}(H - \lambda_1 - i\varepsilon) \Im \mathrm{Tr}(H - \lambda_2 - i\varepsilon)\}$$

In the case of bulk local regime we take $\lambda_{1,2} = E + \xi_{1,2}/\rho(E)\, N$, $E \in (-2, 2)$.

Crossover for the second order correlation function

In the delocalization region (for $d = 1$, when $W >> \sqrt{N}$)

$$(N\rho(E))^{-2} R_2 \left(E + \frac{\xi_1}{\rho(E)\, N}, E + \frac{\xi_2}{\rho(E)\, N} \right) \longrightarrow 1 - \frac{\sin^2(\pi(\xi_1 - \xi_2))}{\pi^2 (\xi_1 - \xi_2)^2},$$

In the localization region (for $d = 1$ when $W << \sqrt{N}$)

$$(N\rho(E))^{-2} R_2 \left(E + \frac{\xi_1}{\rho(E)\, N}, E + \frac{\xi_2}{\rho(E)\, N} \right) \longrightarrow 1,$$

Previous results: $d = 1$

- Fyodorov, Mirlin (1991) – existence of the crossover for $W^2 \sim N$ (on the level of rigour of theoretical physics)
- Schenker (2009) $\ell \leq W^8$ – localization techniques;
- Erdős, Yau, Yin (2011) $\ell \geq W$ – RM methods;
- Erdős, Knowles (2011): $\ell \gg W^{7/6}$;
- Erdős, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$;
- T.Shcherbina (2013): GUE statistics for Wegner band matrix (fixed n);
- Bourgade, Erdős, Yau, Yin (2016) GUE statistics for $W \sim n$.

- S.Sodin (2010): Edge universality iff $W >> N^{5/6}$

Main objects

"Generalised" correlation functions

$$\mathcal{R}_1(z_1, z_1') := \mathbb{E}\Big\{\frac{\det(H - z_1')}{\det(H - z_1)}\Big\}$$

$$\mathcal{R}_2(z_1, z_1'; z_2, z_2') := \mathbb{E}\Big\{\frac{\det(H - z_1')\det(H - z_2'))}{\det(H - z_1)\det(H - z_2))}\Big\}$$

We study these functions for $z_{1,2} = E + \xi_{1,2}/\rho(E)N$, $z_{1,2}' = E + \xi_{1,2}'/\rho(E)N$

Link with the spectral correlation functions:

$$E\{\mathrm{Tr}(H - z_1)^{-1}\mathrm{Tr}(H - z_2)^{-1}\} = \frac{d^2}{dz_1' dz_2'}\mathcal{R}(z_1, z_1'; z_2, z_2')\Big|_{z_1'=z_1, z_2'=z_2}$$

Correlation function of the characteristic polynomials:

$$\mathcal{R}_0(\lambda_1, \lambda_2) = \mathbb{E}\Big\{\det(H - \lambda_1)\det(H - \lambda_2)\Big\}, \quad \lambda_{1,2} = \lambda_0 \pm \xi/n.$$

Integral representations for $\mathcal{R}_{0,1,2}$

There are a scalar kernel $\mathcal{K}_0(X_1, X_2)$, 2×2 matrix kernel $\mathcal{K}_1(X_1, X_2)$, and 70×70 matrix kernel $\mathcal{K}_2(X_1, X_2)$ (containing $z_{1,2}, z'_{1,2}$ as parameters) such that

$$\mathcal{R}_0(\lambda_1,\lambda_2) = C_N \int g_0(X_1)\mathcal{K}_0(X_1,X_2)\dots\mathcal{K}_0(X_{n-1},X_n)f_0(X_n)\prod dX_i,$$

$$X_j = (x_j, y_j, U_j),\ x_j, y_j \in \mathbb{R},\ U_j \in \mathring{U}(2)$$

$$\mathcal{R}_1(z_1,z'_1) = W^2 \int g_1(X_1)\mathcal{K}_1(X_1,X_2)\dots\mathcal{K}_1(X_{n-1},X_n)f_1(X_n)\prod dX_i,$$

$$X_j = (x_j, y_j), \quad x_j, y_i \in \mathbb{R},$$

$$\mathcal{R}_2(z_1,z'_1;z_2,z'_2) = W^4 \int g_2(X_1)\mathcal{K}_2(X_1,X_2)\dots\mathcal{K}_2(X_{n-1},X_n)f_2(X_n)\prod dX_i$$

$$X_j = (x_j, y_j, U_j, S_j,),\ x_j, y_j \in \mathbb{R}^2,\ U_j \in \mathring{U}(2),\ S_j \in \mathring{U}(1,1)$$

dX means an integration over the Haar measure of X,

Recall that the hyperbolic matrix S satisfies the relation

$$S \in \mathring{U}(1,1) \quad \Leftrightarrow \quad S^*LS = L, \quad L = \mathrm{diag}\{1,-1\}$$

Idea of the transfer operator approach

Observation

Let $\mathcal{K}(X,Y)$ be the p-dimensional matrix kernel of the compact integral operator in $\oplus_{i=1}^{p} L_2[X, d\mu(X)]$. Then

$$\int g(X_1)\mathcal{K}(X_1,X_2)\dots\mathcal{K}(X_{n-1},X_n)f(X_n)\prod d\mu(X_i) = (\mathcal{K}^{n-1}f,\bar{g})$$
$$= \sum_{j=0}^{\infty}\lambda_j^{n-1}(\mathcal{K})c_j, \quad \text{with} \quad c_j = (f,\psi_j)(g,\tilde{\psi}_j) \tag{1}$$

Here $\{\lambda_j(\mathcal{K})\}_{j=0}^{\infty}$ are the eigenvalues of $\mathcal{K}$ ($|\lambda_0| \geq |\lambda_1| \geq \dots$), ψ_j are corresponding eigenvectors and $\tilde{\psi}_j$ are the eigenvectors of $\mathcal{K}^*$

Main technical problems

- $\mathcal{K}_{0,1,2}$ are not self adjoint operators, hence we can not use a standard perturbation theory;
- $\mathcal{R}_0$ contains the integration over unitary group $\mathrm{U}(2)/\mathrm{U}(1)\times\mathrm{U}(1)$, and $\mathcal{R}_2$, contains the integration over unitary and hyperbolic $(\mathrm{U}(1,1)/\mathrm{U}(1)\times\mathrm{U}(1))$ groups, hence we need to work with corresponding special functions;
- $\mathcal{K}_1$ is a 2×2 matrix kernel, containing the Jordan cell, and $\mathcal{K}_2$ is a $2^8\times 2^8$ matrix kernel, containing 4×4 Jordan cell in the main block. Using the symmetry of the problem, $\mathcal{K}_2$ could be replaced by 70×70 matrix kernel.

Resolvent version of the transfer operator approach

Observation 2

$$(\mathcal{K}^n f, \bar{g}) = -\frac{1}{2\pi i}\oint_L z^n (\mathcal{G}(z) f, \bar{g}) dz, \quad \mathcal{G}(z) = (\mathcal{K} - z)^{-1}$$

where L is any closed contour which contains all eigenvalues of $\mathcal{K}$. It is sufficient to take $L = L_0 = \{|z| = 1 + Cn^{-1}\}$,

We choose $L = L_1 \cup L_2$ where $L_2 = \{z : |z| = 1 - \log^2 n/n\}$, and L_1 is some special contour, containing all eigenvalues between L_0 and L_2. Then

$$(\mathcal{K}_\alpha^n f, \bar{g}) = -\frac{1}{2\pi i}\oint_{L_1} z^n (\mathcal{G}_\alpha(z) f, \bar{g}) dz - \frac{1}{2\pi i}\oint_{|z|=1-\log^2 n/n} z^n (\mathcal{G}_\alpha(z) f, \bar{g}) dz$$

The second integral is small since $|z|^n \le e^{-\log^2 n}$

Definition of asymptotically equivalent operators $(n, W \to \infty)$

$$\mathcal{A}_{Wn} \sim \mathcal{B}_{Wn} \quad \Leftrightarrow \oint_{L_1} z^n ((\mathcal{A}_{Wn} - z)^{-1} f, \bar{g}) dz = \oint_{L_1} z^n ((\mathcal{B}_{Wn} - z)^{-1} f, \bar{g}) dz + o(1)$$

Mechanism of the crossover for $\mathcal{R}_0$

Key technical step

$$\mathcal{K}_{0\xi} \sim \mathcal{K}_{*\xi} \otimes \mathcal{A},$$

$$\mathcal{K}_{*\xi}(\mathrm{U}_1, \mathrm{U}_2) = \mathrm{e}^{-\mathrm{i}\xi\nu(\mathrm{U}_1)/\mathrm{N}}\mathrm{K}_*(\mathrm{U}_1\mathrm{U}_2^*)\mathrm{e}^{-\mathrm{i}\xi\nu(\mathrm{U}_2)/\mathrm{N}}, \quad \mathcal{K}_{0\xi} : \mathrm{L}_2(\mathring{\mathrm{U}}(2)) \to \mathrm{L}_2(\mathring{\mathrm{U}}(2)),$$

$$\mathcal{A}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2) = \mathrm{A}_1(\mathrm{x}_1, \mathrm{x}_2)\mathrm{A}_2(\mathrm{y}_1, \mathrm{y}_2), \qquad \mathrm{L}_2(\mathbb{R}^2) \to \mathrm{L}_2(\mathbb{R}^2).$$

Here $\xi_1 = -\xi_2 = \xi$, and $\nu(\mathrm{U}) = \pi(1 - 2|\mathrm{U}_{12}|^2)$

Then

$$\mathcal{R}_0 = (\mathcal{K}_{*\xi}^{\mathrm{N}} \otimes \mathcal{A}^{\mathrm{N}}\mathrm{f}, \bar{\mathrm{g}})(1 + o(1)) = (\mathcal{K}_{*\xi}^{\mathrm{N}}\mathrm{f}_0, \mathrm{f}_0)(\mathcal{A}^{\mathrm{N}}\mathrm{f}_1, \bar{\mathrm{g}}_1)(1 + o(1)).$$

Here we used that both f, g asymptotically can be replaced by $\mathrm{f}_0(\mathrm{U}) \otimes \mathrm{f}_1(\mathrm{x}, \mathrm{y})$ ($\mathrm{f}_0 = 1$). If we introduce the normalization constant

$$\mathrm{D}_2 = \mathcal{R}_0(\mathrm{E}, \mathrm{E}).$$

then

$$\mathrm{D}_2^{-1}\mathcal{R}_0\Big(\mathrm{E} + \frac{\xi}{\mathrm{N}\rho(\mathrm{E})}, \mathrm{E} - \frac{\xi}{\mathrm{N}\rho(\mathrm{E})}\Big) = \frac{(\mathcal{K}_{*\xi}^{\mathrm{N}}\mathrm{f}_0, \mathrm{f}_0)}{(\mathcal{K}_{*0}^{\mathrm{N}}\mathrm{f}_0, \mathrm{f}_0)}(1 + o(1))$$

Spectral analysis of $\mathcal{K}_{*\xi}$

A good news is that $\mathcal{K}_{*0}$ with a kernel

$$\mathcal{K}_{*0} = \mathrm{t}_*\mathrm{W}^2 \mathrm{e}^{-\mathrm{t}_*\mathrm{W}^2|(\mathrm{U}_1\mathrm{U}_2^*)_{12}|^2}$$

is a self-adjoint "difference" operator. It is known that his eigenfunctions are Legendre polynomials P_j. Moreover, it is easy to check that corresponding eigenvalues have the form:

$$\lambda_\mathrm{j} = 1 - \mathrm{t}_*\mathrm{j}(\mathrm{j}+1)/\mathrm{W}^2 + \mathrm{O}((\mathrm{j}(\mathrm{j}+1)/\mathrm{W}^2)^2), \quad \mathrm{j} = 0, 1 \ldots.$$

Besides,

$$\mathcal{K}_{*\xi} = \mathcal{K}_{*0} - 2\mathrm{i}\xi\hat{\nu}/\mathrm{N} + \mathrm{O}(\mathrm{N}^{-2})$$

where $\hat{\nu}$ is the operator of multiplication by ν. Thus the eigenvalues of $\mathcal{K}_{*\xi}$ are in the N^{-1}-neighbourhood of λ_j.

Mechanism of the Poisson behavior for $W^2 << N$

For $W^{-2} >> N^{-1}$ (the spectral gap is much less then the perturbation norm)

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 - 2N^{-1} i\xi(\nu f_0, f_0) + o(N^{-1}),$$
$$|\lambda_1(\mathcal{K}_{*\xi})| \le 1 - O(W^{-2}) \quad \Rightarrow \quad |\lambda_j(\mathcal{K}_{*\xi})|^N \to 0, \ (j = 1, 2, \dots).$$

Since

$$(\nu f_0, f_0) = 0,$$

we obtain that

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 + o(N^{-1}),$$

and

$$D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{\lambda_0^N(\mathcal{K}_{*\xi})}{\lambda_0^N(\mathcal{K}_{*0})}(1 + o(1)) \to 1$$

The relation corresponds to the Poisson local statistics.

Mechanism of the GUE behavior for $W^2 >> N$

In the regime $W^{-2} << N^{-1}$ we have $\mathcal{K}_{*0}^N \to I$ in the strong vector topology, hence one can prove that

$$\mathcal{K}_{*\xi} \sim 1 + O(W^{-2}) - N^{-1} 2i\xi\nu \Rightarrow (\mathcal{K}_{*\xi}^N f_0, f_0) \to (e^{-2i\xi\hat{\nu}} f_0, f_0)$$

and

$$D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(e^{-2i\xi t^*\hat{\nu}} f_0, f_0)}{(f_0, f_0)}(1 + o(1)) \to \frac{\sin(2\pi\xi)}{2\pi\xi}.$$

The expression for $D_2^{-1}\mathcal{R}_0$ coincides with that for GUE.

In the regime $W^{-2} = C_* N^{-1}$ observe that $\mathcal{K}_{*\xi}$ is reduced by the subspace $\mathcal{E}_0$ of the functions depending only on $|U_{12}|^2$.
Recall also that the Laplace operator on $\mathring{U}(2)$ is reduced by $\mathcal{E}_0$ and have the form

$$\Delta_U = -\frac{d}{dx} x(1-x)\frac{d}{dx}, \quad x = |U_{12}|^2.$$

Besides, the eigenvectors of Δ_U and $\mathcal{K}_{*0}$ coincide (they are Legendre's polynomials P_j) and corresponding eigenvalues of Δ_U are

$$\lambda_j^* = j(j+1).$$

Hence we can write $\mathcal{K}_{*\xi}$ as

$$\mathcal{K}_{*\xi} \sim 1 - N^{-1}(C_* t_* \Delta_U + 2i\xi\nu) + o(N^{-1}) \Rightarrow (\mathcal{K}_{*\xi}^N f_0, f_0) \to (e^{-C\Delta_U - 2i\xi\hat{\nu}} f_0, f_0)$$

Results for $\mathcal{R}_0$

Theorem 1 [TS:14]

$N < W^{2-\theta}$, where $0 < \theta < 1$, and $E \in (-2,2)$ we have

$$\lim_{N,W\to\infty} D_2^{-1}\mathcal{R}_0\Big(\lambda_0 + \frac{\xi}{N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{N\rho(\lambda_0)}\Big) = \frac{\sin(2\pi\xi)}{2\pi\xi},$$

i.e. the limit coincides with that for GUE. The limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$. Here

$$D_2 = \mathcal{R}_0(\lambda_0, \lambda_0).$$

Theorem 2 [TS,MS:16]

$N > CW^2 \log W$

$$\lim_{N,W\to\infty} D_2^{-1}\mathcal{R}_0\Big(\lambda_0 + \frac{\xi}{N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{N\rho(\lambda_0)}\Big) = 1$$

The limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$.

Theorem 3 [TS: in preparation]

For 1d RBM with $N = C_* W^2$, $E \in (-2, 2)$, we have

$$\lim_{N\to\infty} D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = (e^{-C_* t_* \Delta_U - 2i\xi\hat{\nu}} f_0, f_0),$$

where $t^* = (2\pi\rho(E))^2$, and the limit is uniform in ξ varying in any compact subset of $\mathbb{R}$.

Result for $\mathcal{R}_1$

$$\mathcal{K}_1 \sim F(x_1, y_1) A_1(x_1, x_2) A_2(y_1, y_2) F(x_2, y_2) \begin{pmatrix} 1 + L(\bar{x}, \bar{y})/W^2 & -1 \\ -L(\bar{x}, \bar{y}) & 1 \end{pmatrix}$$

Operators A_1 and A_2 contain a large parameter W in the exponent, hence only $W^{-1/2}$ neighbourhood of the stationary point (x^*, y^*) gives essential contribution. The function $L(\bar{x}, \bar{y})$ here satisfies the relation

$$L(\bar{x}, \bar{y}) = 0 \Big|_{\bar{x}=\bar{x}^*, \bar{y}=\bar{y}^*}$$

Hence the main order of our operator contains the Jordan cell.
The spectral gap of $A_{1,2}$ is $O(W^{-1}) >> N^{-1}$, hence $A_{1,2}^N \sim \lambda_0^N(A_{1,2}) P_{1,2}$ $(\operatorname{rank} P_{1,2} = 1)$

Theorem 3 [MS,TS:16]

Let $N \geq C_0 W \log W$, and $|\lambda_0| \leq 4\sqrt{2}/3 \approx 1.88$. Then we have for the first correlation function R_1 (the first marginal density)

$$|R_1(E) - \rho(E)| \leq C/W$$

Sigma-model $\mathcal{R}_2^{(\sigma)}$

The model can be obtained by some scaling limit ($\alpha = \beta/\mathrm{W}$, $\mathrm{W} \to \infty$, β, n-fixed) from the expression for $\mathcal{R}_2$.
The crossover is expected for $\beta \sim \mathrm{n}$.

$$\mathcal{R}_2^{(\sigma)} = \int \exp\left\{\frac{\beta}{4}\sum \mathrm{Str}\, \mathrm{Q_j Q_{j+1}} + \frac{\varepsilon + i\xi}{4\mathrm{n}} \sum \mathrm{Str}\, \mathrm{Q_j}\Lambda\right\}$$
$$\times \prod (1 - 2\rho_{1j}\tau_{1j}\rho_{2j}\tau_{2j}) \prod d\mathrm{Q_j}$$

Here $\mathrm{Q_j}$ is a 4×4 super matrix of the block form:

$$\mathrm{Q_j} = \begin{pmatrix} \mathrm{U_j^*} & 0 \\ 0 & \mathrm{S_j^{-1}} \end{pmatrix} \begin{pmatrix} \mathrm{T_{1j}} & \mathrm{T_{2j}} \\ \mathrm{T_{3j}} & \mathrm{T_{4j}} \end{pmatrix} \begin{pmatrix} \mathrm{U_j} & 0 \\ 0 & \mathrm{S_j} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \mathrm{L} & 0 \\ 0 & -\mathrm{L} \end{pmatrix}$$

$\mathrm{T_{1j}} = \mathrm{diag}\{1 + 2\rho_{1j}\tau_{1j}; -(1 + 2\rho_{2j}\tau_{2j})\}$ $\quad \mathrm{T_{4j}} = \mathrm{diag}\{-1 + 2\rho_{1j}\tau_{1j}; 1 - 2\rho_{2j}\tau_{2j})\}$
$\mathrm{T_{2j}} = \mathrm{diag}\{2\tau_{1j}; 2\tau_{2j}\}$; $\quad \mathrm{T_{3j}} = \mathrm{diag}\{2\rho_{1j}; 2\rho_{2j}\}$

Here $\{\mathrm{U_j}\}_{j=1}^{n}$ are unitary matrices, $\{\mathrm{S_j}\}_{j=1}^{n}$ are hyperbolic matrices and

$$d\mathrm{Q_j} = d\mathrm{U_j} d\mathrm{S_j} d\rho_{1j} d\rho_{2j} d\tau_{1j} d\tau_{2j}$$

Transfer operator for $\mathcal{R}_2^{(\sigma)}$

The kernel of the transfer operator for $\mathcal{R}_2^{(\sigma)}$ has a form

$$\mathcal{K}_2^{(\sigma)} = \hat{\mathrm{F}}\hat{\mathrm{Q}}\hat{\mathrm{F}}$$

where $\hat{\mathrm{F}}$ and $\hat{\mathrm{Q}}$ are 6×6 matrix kernels, s.t. $\hat{\mathrm{F}}_{\mu\nu}$ are multiplication operators and $\hat{\mathrm{Q}}_{\mu\nu} = \hat{\mathrm{Q}}_{\mu\nu}(\mathrm{U}_1\mathrm{U}_2^*, \mathrm{S}_1\mathrm{S}_2^{-1})$ are "difference" operators.
The key step is to prove

$$\hat{\mathrm{F}}\hat{\mathrm{Q}}\hat{\mathrm{F}} \sim \tilde{\mathrm{F}}\hat{\mathrm{K}}_0\tilde{\mathrm{F}},$$

where $\hat{\mathrm{K}}_0$ and $\tilde{\mathrm{F}}$ are 4×4 matrices of the form

$$\hat{\mathrm{K}}_0 = \begin{pmatrix} \mathrm{K} & \tilde{\mathrm{K}}_1 & \tilde{\mathrm{K}}_2 & \tilde{\mathrm{K}}_3 \\ 0 & \mathrm{K} & 0 & \tilde{\mathrm{K}}_2 \\ 0 & 0 & \mathrm{K} & \tilde{\mathrm{K}}_1 \\ 0 & 0 & 0 & \mathrm{K} \end{pmatrix}, \quad \tilde{\mathrm{F}} = \mathrm{F}\begin{pmatrix} 1 & \tilde{\mathrm{F}}_1 & \tilde{\mathrm{F}}_2 & \tilde{\mathrm{F}}_1\tilde{\mathrm{F}}_2 \\ 0 & 1 & 0 & \tilde{\mathrm{F}}_2 \\ 0 & 0 & 1 & \tilde{\mathrm{F}}_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\mathrm{K} = \mathrm{K}_\mathrm{U} \otimes \mathrm{K}_\mathrm{S}$

$$\mathrm{K}_\mathrm{U}(\mathrm{U}_1, \mathrm{U}_2) \sim \beta e^{-\beta|(\mathrm{U}_1\mathrm{U}_2^*)_{12}|^2}, \ \mathrm{K}_\mathrm{S}(\mathrm{S}_1, \mathrm{S}_2) \sim \beta e^{-\beta|(\mathrm{S}_1\mathrm{S}_2^{-1})_{12}|^2}$$

$\tilde{\mathrm{K}}_\mathrm{i} = \tilde{\mathrm{K}}_\mathrm{i}(\mathrm{U}_1\mathrm{U}_2^*; \mathrm{S}_1\mathrm{S}_2^{-1}) \sim \beta^{-1}\Delta_{\mathrm{U},\mathrm{V}}$, F is an operator of multiplication by $e^{\varphi(\mathrm{U},\mathrm{S})/2n}$, and $\tilde{\mathrm{F}}_{1,2}$ are operators of multiplication by $n^{-1}\varphi_{1,2}(\mathrm{U}, \mathrm{S})$

Result for $\mathcal{R}_2^{(\sigma)}$

Theorem 4 [MS,TS:17] (submitted to JSP)

For the sigma-model in the regime $C\beta/\log^2\beta > n$

$$\lim_{n\to\infty} \mathcal{R}_2^{(\sigma)} = (\hat{F}_0\tilde{f}, \tilde{g})$$

where

$$\hat{F}_0 = F_0 \begin{pmatrix} 1 & F_1 & F_2 & F_1F_2 \\ 0 & 1 & 0 & F_2 \\ 0 & 0 & 1 & F_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F_0 \sim e^{\varphi(U,S)}, \quad F_{1,2} \sim \varphi_{1,2}(U,S)$$

$$\tilde{f} = (e_4 - e_1), \quad \tilde{g} = (e_1 - e_4)$$

Corollary

For $|E| \le \sqrt{2}$ the second order correlation function of RBBM with $\alpha = \beta/W$ in the limit $W \to \infty$ and then $\beta, n \to \infty$, $(\beta >> n)$ coincides with that for GUE.

Transfer operator for $\mathcal{R}_2$

The kernel of the transfer operator for $\mathcal{R}_2$ has a form

$$\mathcal{K}_2 = \hat{\mathrm{F}}\hat{\mathrm{Q}}\hat{\mathrm{A}}\hat{\mathrm{F}}$$

where $\hat{\mathrm{F}}$, $\hat{\mathrm{Q}}$ and $\hat{\mathrm{A}}$ are 70×70 matrix kernels, s.t. $\hat{\mathrm{F}}_{\mu\nu}$ are the operators of multiplication by some function of U, S,

$$\hat{\mathrm{Q}}_{\mu\nu} = \mathrm{K_U K_S Q}_{\mu\nu}(\mathrm{U_1 U_2^*; S_1 S_2^{-1}}),$$
$$\mathrm{K_U} = \alpha \mathrm{W} e^{-\alpha \mathrm{W} t(\bar{x},\bar{y})|(\mathrm{U_1 U_2^*})_{12}|^2}, \quad \mathrm{K_S} = \alpha \mathrm{W} e^{-\alpha \mathrm{W} t(\bar{x},\bar{y})|(\mathrm{S_1 S_2^{-1}})_{12}|^2},$$

and

$$\hat{\mathrm{A}}_{\mu\nu} = \mathrm{A}_1(x_1, x_2)\mathrm{A}_2(y_1, y_2)\mathrm{A}_3(x_1', x_2')\mathrm{A}_4(y_1', y_2')\mathcal{A}_{\mu,\nu}(\bar{x}, \bar{y})$$
$$\mathrm{A}_\delta(x, y) = (\alpha \mathrm{W}/2\pi)^{1/2} e^{-\mathrm{W}\alpha(x-y)^2/2 + \mathrm{W}(f_\delta(x) + f_\delta(y))}, \quad \delta = 1, 2, 3, 4.$$

The spectral gap for A_δ is $\mathrm{O}(1)$.

Result for $\mathcal{R}_2$

After a rather involved asymptotic analysis we obtain

$$\mathcal{K}_2 \sim \tilde{\mathrm{F}}\hat{\mathrm{K}}_0\tilde{\mathrm{F}}$$

where $\hat{\mathrm{K}}_0$ and $\tilde{\mathrm{F}}$ are 4×4 matrices similar to that for sigma-model.

Theorem 5 [MS,TS:17] (in preparation)

For $|\mathrm{E}| \le \sqrt{2}$ and $\mathrm{W}^2/\log^2 \mathrm{W} > \mathrm{CN}$,

$$\lim_{n\to\infty} \mathcal{R}_2 = (\hat{\mathrm{F}}_0\tilde{\mathrm{f}}, \tilde{\mathrm{g}})$$

where $\hat{\mathrm{F}}_0, \tilde{\mathrm{f}}, \tilde{\mathrm{g}}$ are the same as in Theorem 4.

Corollary

For $|\mathrm{E}| < \sqrt{2}$ the second order correlation function of 1d RBBM in the limit $\mathrm{N}, \mathrm{W} \to \infty$, $\mathrm{W}^2/\log^2 \mathrm{W} > \mathrm{CN}$, coincides with that for GUE.