Large block asymptotics of entanglement entropy of free fermions

L. Pastur

ILTPE Kharkiv Ukraine

Joint work with V. Slavin and M. Shcherbina

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Outline

- Entanglement Entropy
- Area Law for Translation Invariant Systems
- Free Fermions
- Area Law for Disordered Fermions
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1 Entanglement Entropy

1.1 Entanglement

Entanglement: a complex and delicate quantum phenomenon (since 1930' *Einstein et al, Schrödinger*), in particular a widely believed resource of quantum informatics (since 1980' *Feynman*).

Bipartite quantum system consists of two parties A(lice) and B(ob), thus has the state space $\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Pure state $\Psi \in \mathcal{H}_{A+B}$ is **entangled** if it is not **separable**

$$\Psi \neq \Psi_A \otimes \Psi_B, \ \Psi_{A,B} \in \mathcal{H}_{A,B}.$$

Example: **Bell states.** A and B are qubits, i.e., $\dim \mathcal{H}_{A,B} = 2$, $(|1\rangle_A, |2\rangle_A)$ and $(|1\rangle_B, |2\rangle_B)$ are respective orthonormal bases and

$$\Psi = 2^{-1/2} \left(|1\rangle_A \otimes |2\rangle_B \pm |2\rangle_A \otimes |1\rangle_B \right)$$

1.2 Entanglement Entropy

Entropy: (i) classical physics: a measure of the lack of knowledge (e.g., on microstates corresponding a given macrostate), hence related to classical probability or randomness (Boltzmann, Gibbs, Shannon)

(ii) quantum physics: a measure of quantum correlations due to the ”randomness” of quantum mechanics given by

von Neumann entropy of a state (density matrix) ρ

$$S(\rho) = -\mathrm{Tr}\rho \log_2 \rho.$$

Simple important properties:

(i) $\rho = |\Psi\rangle \langle\Psi| \Leftrightarrow S(\rho) = 0$

(ii) $\dim \mathcal{H} = n \Leftrightarrow \max_\rho S(\rho) = S(\mathbf{1}_n/n) = \log_2 n,$

cf. classical case where entropy is zero for degenerate discrete distribution and is maximal for the uniform discrete distribution.

Reduced Density Matrix If ρ is a density matrix (”joint probability distribution”) of a bipartite system $A+B$, then (*Dirac, Landau*)

$$\rho_A = \mathrm{Tr}_B \rho$$

is the **reduced density matrix** (q-analog of the marginal distribution) of A.

Entanglement entropy of A:

$$S(\rho_A) = -\mathrm{Tr}_A\, \rho_A \log_2 \rho_A.$$

Example: If Ψ is the Bell state, then $S(|\Psi\rangle\langle\Psi|) = 0$ (valid for any pure state). However,

$$\rho_A = 2^{-1}(|1\rangle_A \langle 1|_A + |2\rangle_A \langle 2|_A) = \mathbf{1}_A/2$$

and $S(\rho_A) = \log_2 2 = 1$, i.e., is maximal possible, i.e., $S(\rho_A)$ is an **entanglement quantifier**.

2 Extended Systems

2.1 Area Law

Consider a quantum bipartite system $A + B$ confined to the d-dimensional domain (cube) $\Omega \subset \mathbb{Z}^d$ of side length N with A (block) confined to a subcube $\Lambda \subset \Omega$ of side length L and B (environment) confined to $\Omega \setminus \Lambda$. One is interested in the asymptotics of $S(\rho_\Lambda)$ for

$$1 << L << N.$$

According to *Bekenstein 1973, Hawking 1974* (black holes physics),
Bombelli et al. 1986, Srednicki 1993, Callan-Wilczek, 1994 (QFT),
Calabrese-Cardy 2005- (CFT, quantum spin chains)

$$S(\rho_\Lambda) \simeq \begin{cases} L^{d-1}, & area\ law, \\ L^{d-1}\log L, & violation\ of\ area\ law. \end{cases}$$

Recently: The area law is valid ”generically” for locally interacting quantum systems having a gap in their spectrum (*Hastings 2010s*).

Violation of the Area Law

(i) $d = 1$: explicitly solvable models, e.g., 1d quantum spin chains, at qpt's (critical regimes).

(ii) $d > 1$: mostly conjectured, established only for the toy model of free translation invariant fermions, i.e., quadratic quantum Hamiltonians

$$\widehat{H} = \sum_{j,k \in \Omega} H_{jk} c_j^+ c_k$$

where $\{c_j, c_k^+\} = \delta_{jk}$ are the Fermi annigilation and creation operators, $H_\Omega := \{H_{jk}\}_{j,k\in\Omega}, H_\Omega = H_\Omega^*$ is the **one body Hamiltonian** of free fermions.

Translation invariant case: $H_\Omega = H|_\Omega$ and $H = \{H_{j-k}\}_{j,k\in\mathbb{Z}^d}$, i.e., H is a convolution operator in $l^2(\mathbb{Z}^d)$ and $H \geq 0$ (without loss of generality).

2.2 Free Fermions

It is standard to show that if $\mathcal{E}_{H_\Omega}$ is he resolution of identity of H_Ω and

$$P_\Omega = \mathcal{E}_{H_\Omega}([0, E]),\ E \in \sigma(H_\Omega)$$

is its **Fermi projection** with **Fermi energy** E, then

$$S_{\Lambda,\Omega} = \mathrm{Tr}_{\ \Lambda}\, h(P_{\Lambda,\Omega}),\ P_{\Lambda,\Omega} = P_\Omega|_\Lambda$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x),\ 0 \le x \le 1,$$

and, assuming that H is ergodic, that

$$\lim_{\Omega\to\mathbb{Z}^d} S_{\Lambda,\Omega} = S_\Lambda == \mathrm{Tr}_{\ \Lambda}\, h(P_\Lambda),\ P_\Lambda = P|_\Lambda$$

We are interested in asymptotic behaviour of S_Λ as $\Lambda \to \mathbb{Z}^d$.

3 Entanglement Entropy of Disordered Fermions

3.1 No Volume Law in the Mean

Theorem 1. *Let* $S_\Lambda = \mathrm{Tr}\ h(P_\Lambda)$ *be the entanglement entropy of free fermions having the discrete Schrödinger operator with ergodic potential as one body Hamiltonian. Then we have for* $\Lambda = [-M, M]^d,\ L = 2M + 1$

$$\lim_{L\to\infty} L^{-d}\mathbf{E}\{S_\Lambda\} = 0.$$

Mere ergodicity of potential excludes the ”volume law”.

3.2 Exponential Decay of Fermi Projection

Let H be the discrete Schrödinger operator with ergodic potential and $P = \{P_{jk}\}_{j,k \in \mathbb{Z}^d}$ be its Fermi projection. The following **exponential bound**:

$$\mathbf{E}\{|P_{jk}|\} \leq Ce^{-\gamma|j-k|},\ C < \infty, \gamma > 0.$$

is valid for the Schrödinger operator in $l^2(\mathbb{Z}^d)$ with

- any potential if the Fermi energy is in a spectral gap;
- random weakly correlated potential having sufficiently regular probability distribution;
- certain classes of quasiperiodic potentials (Diophantine frequencies).

The first case is quite general and the last two cases correspond to the existence of pure point spectrum with exponentially decaying eigenfunctions (**Anderson localization**).

For the both classes (random and quasiperiodic) the bound is valid for (examples):

- any $E \in \sigma_H$, $d = 1$ and i.i.d. potentials (*Minami 96*);
- any $E \in \sigma_H$, $d \geq 1$ and i.i.d. potentials, of a sufficiently large amplitude or E belong1ng to a neighborhood of the spectrum edges, $d \geq 1$ and i.i.d. potentials of a fixed amplitude (*Aizenman-Molchanov 93, Aizenman-Graf 98*);
- any E, $d = 1$, and quasiperiodic potentials of sufficiently large amplitude (*Jitomirskaya et al 1998 – 2013*).

3.3 Area Law in the Mean.

Theorem 2. *Let* $S_\Lambda = \mathrm{Tr}\ h(P_\Lambda)$ *be the entanglement entropy of free disordered fermions whose one body Hamiltonian is the discrete Schrödinger operator* H *with ergodic potential. Assume that its Fermi projection admits the exponential bound. Then we have for* $\Lambda = [-M, M]^d$*,* $L = 2M + 1$*:*

$$\mathbf{E}\{S_\Lambda\} = \sigma_d L^{d-1}(1 + o(1)),\ L \to \infty,$$
$$\sigma_d = 2d\, \mathbf{E}\{\mathrm{Tr}\ h(P_{\mathbb{Z}^d_+})\}.$$

Unlike the convolution operators with the absolutely continuous (ballistic) spectrum leading to the violation of the area law, ergodic operators with pure point spectrum lead to the area law, although in the mean.

No the violation of the area law in the gapless spectrum!, cf. Hastigs's theorem

3.4 Entanglement Entropy is Selfaveraging for $d \geq 2$.

Theorem 3. *Let* $S_\Lambda = \{Trh(P_\Lambda)$ *be the entanglement entropy of free disordered fermions whose one body Hamiltonian is the discrete Schrödinger operator* H *with with i.i.d. random potential. Assume that its Fermi projection and admits the exponential bound. Then there exists* L*-independent* $C_d < \infty$ *and* $a_d > 0$ *such that we have for* $d \geq 2$ *and* $\Lambda = [-M, M]^d,\ L = 2M + 1$

$$\mathbf{Var}\{L^{-(d-1)} S_\Lambda\}$$

$$:= \mathbf{E}\{(L^{-(d-1)} S_\Lambda)^2\} - (\mathbf{E}\{L^{-(d-1)} S_\Lambda\})^2 \leq C_d L^{-a_d}.$$

3.5 Asymptotic Behavior with Probability 1 for $d = 1$

Theorem 4. *Let $S_\Lambda = \mathrm{Tr}\, h(P_\Lambda)$ be the entanglement entropy of disordered fermions for $d = 1$ whose one body Hamiltonian is the discrete Schrödinger operator H with with ergodic random potential. Assume that its Fermi projection admits the exponential bound.*

Then we have with probability 1 for $\Lambda = [-M, M],\ L = 2M + 1$:

$$S_\Lambda(\omega) = S_+(T^{+M}\omega) + S_-(T^{-M}\omega) + o(1)),\ L \to \infty,$$

where

$$S_\pm = \mathrm{Tr}\, h(P_{\mathbb{Z}_\pm}).$$

are not zero with probability 1.

3.6 Ideas of Proofs

(i) **Existence of the limit.** Introduce $h_0 : [0, 1/4] \to [0, 1]$ by

$$h(x) = h_0(x(1-x)),\ h_0 \in C^\alpha,\ \alpha \in (0,1).$$

Hence

$$S_\Lambda := \mathrm{Tr}\ h(P_\Lambda) = \mathrm{Tr}\ h_0(\Pi_\Lambda),\ \Pi_\Lambda = P_\Lambda(1_\Lambda - P_\Lambda),$$

$$(\Pi_\Lambda)_{j_1 j_2} = \sum_{|k|>M} P_{j_1,k} P_{k,j_2},\ |j_1|, |j_2| \le M$$

and because of the exponential decay of $\mathbf{E}\{|P_{jk}|\}$ the entries of Π_Λ are concentrated in a surface layer of Λ.

(ii) **Variance.** Divide the surface layer into large number of ”mesoscopic” pieces of size $l, \; 1 << l << L$ separated by corridors of size $l_1, \; 1 << l_1 << l$ and prove that the contributions of pieces are ”almost” independent.

(iii) Technical issues (h and h_0, are not C^1) are facilitated by the bound (*A.Sobolev 16*)

$$|\,\mathrm{Tr}\;\, h_0(A) - \mathrm{Tr}\;\, h_0(B)| \le C\,\mathrm{Tr}\;\, |A - B|^{\alpha}, \; \alpha \in (0, 1).$$

Translation invariant case. Use $h(x) \geq 4x(1-x)$, $x \in [0,1]$ and $P_{jk} = \sin \kappa (j-k)/\pi(j-k)$ implying

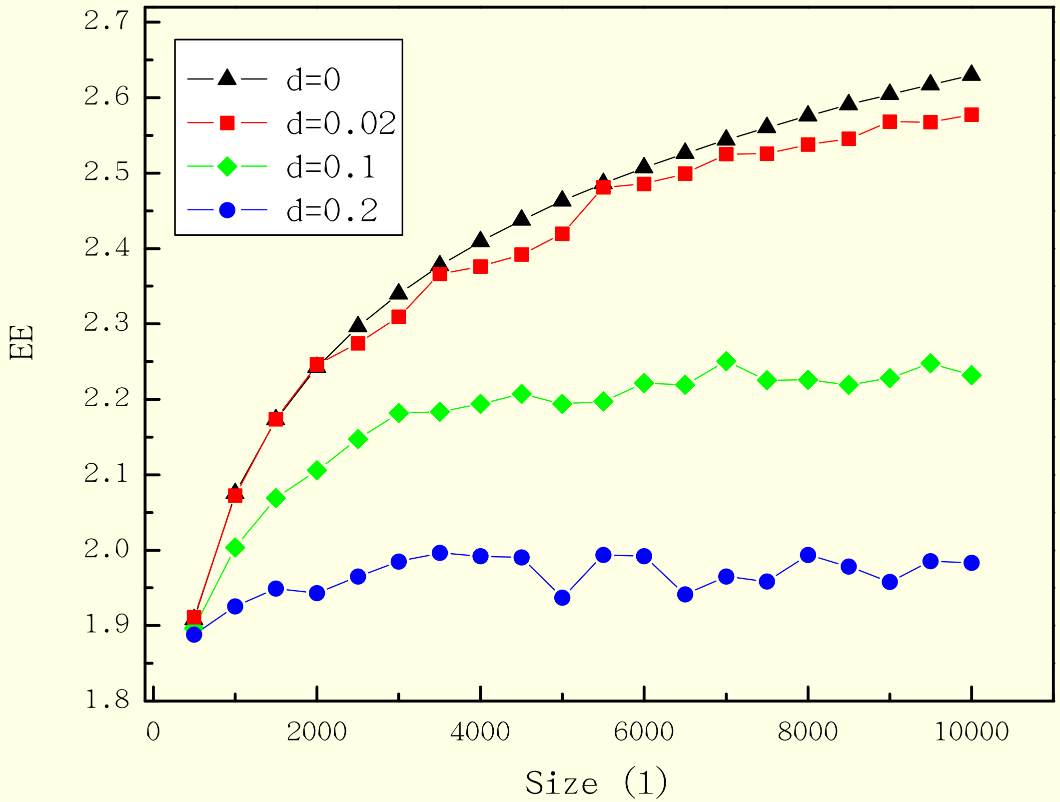
$$S_\Lambda \geq 4 \operatorname{Tr} \Pi_\Lambda = \frac{8}{\pi^2} \sum_{j=0}^{2M} \sum_{k \geq 1} \frac{\sin^2(j+k)}{(j+k)^2} = \frac{8}{\pi^2} \log L + O(1), \ L \to \infty.$$

Likewise, since h_0 is concave

$$S_\Lambda = \sum_{|j| \leq M} (h_0(\Pi_\Lambda))_{jj} \leq \sum_{|j| \leq M} h_0(\Pi_{jj}) \leq C(\log L)^2$$

since $\Pi_{jj} \simeq (M+1-j)^{-1}$ and $h_0(x) = h(x) \simeq -x \log x, \ x \to 0$.

3.7 Emergence of the Area Law for $d = 1$



3.8 Entanglement Entropy is not Selfaveraging for $d = 1$

Theorem 5. *Let* $S_\Lambda = \mathrm{Tr}\, h(P_\Lambda)$ *be the entanglement entropy of disordered fermions for* $d = 1$ *whose one body Hamiltonian is the discrete Schrödinger operator* H *with i.i.d. random potential. Assume that its common probability law has a bounded probability density* f *such that*

(i) $\mathrm{supp} f = [0, \infty)$ *and* $\mathbf{E}\{(V(0))^\kappa\} < \infty$ *for some* $\kappa > 0$*,*

(ii) the quantity $F(t) = J(t) - 1$ *with*

$$J(t) = \int_0^\infty (f^2(v - t)/f(v))dv, \; t \in [0, \infty)$$

is finite for all sufficiently large $t > 0$*.*

Then there exist $t_0 > 0$ *and* $M_0 > 0$ *such that for all* $M > M_0$

$$\mathbf{Var}\{S_{[-M,M]}\} \geq \mathbf{E}^2\{S_-\}/2F(t_0) > 0,$$

where $S_- = \mathrm{Tr}\, h(P_{\mathbb{Z}_-})$ *is not zero with probability 1.*

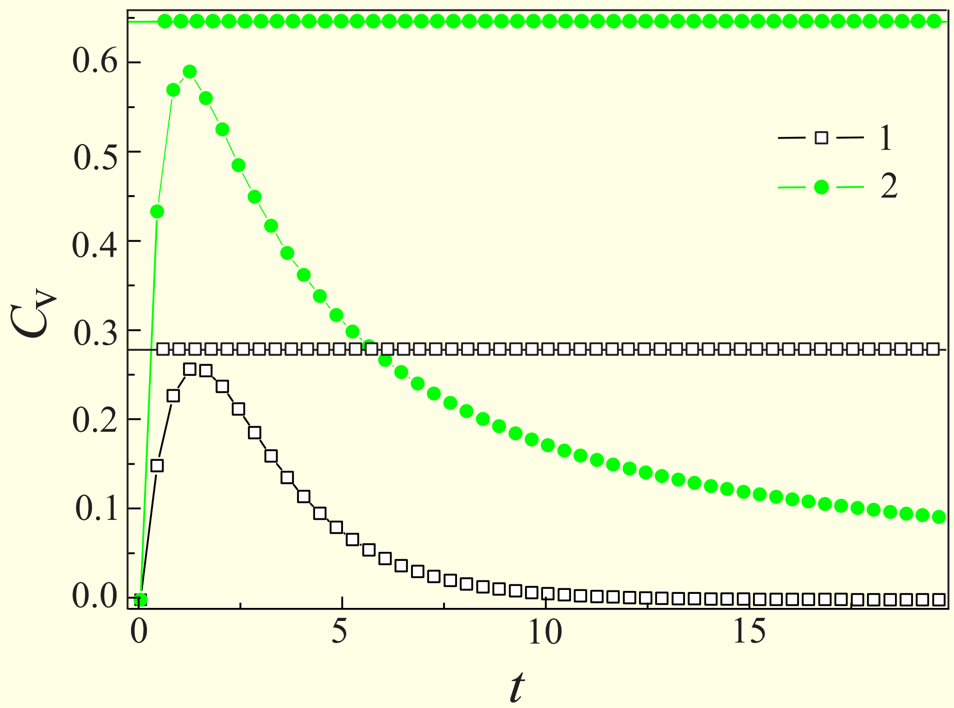
Examples: (i) Exponential

$$f(v) = \delta^{-1} e^{-v/\delta} \mathbf{1}_{[0,\infty)},$$

(ii) ”half”-Cauchy

$$f(v) = \frac{2\delta}{\pi(v^2 + \delta^2)} \mathbf{1}_{[0,\infty)},$$

but not uniform $f(v) = \delta^{-1}\mathbf{1}_{[0,\delta)}$ or ”half”-Gaussian distributions.



The square root $C_{\mathbf{V}}$ of the lower bound of the theorem versus t is quite close to $\mathbf{C_V}\{S_{[-M,M]}\} = (\mathbf{Var}\{S_{[-M,M]}\})^{1/2}/(\mathbf{E}\{S_{[-M,M]}\})$, black for exponential, green for half-Cauchy (the maxima of the plots are about 92 percents of the respective $\mathbf{C_V}\{S_{[-M,M]}\}$, indicated by horizontal lines).

3.9 Idea of proof

Use the Hammersley-Chapman-Robins inequality (à la Cramér-Rao)

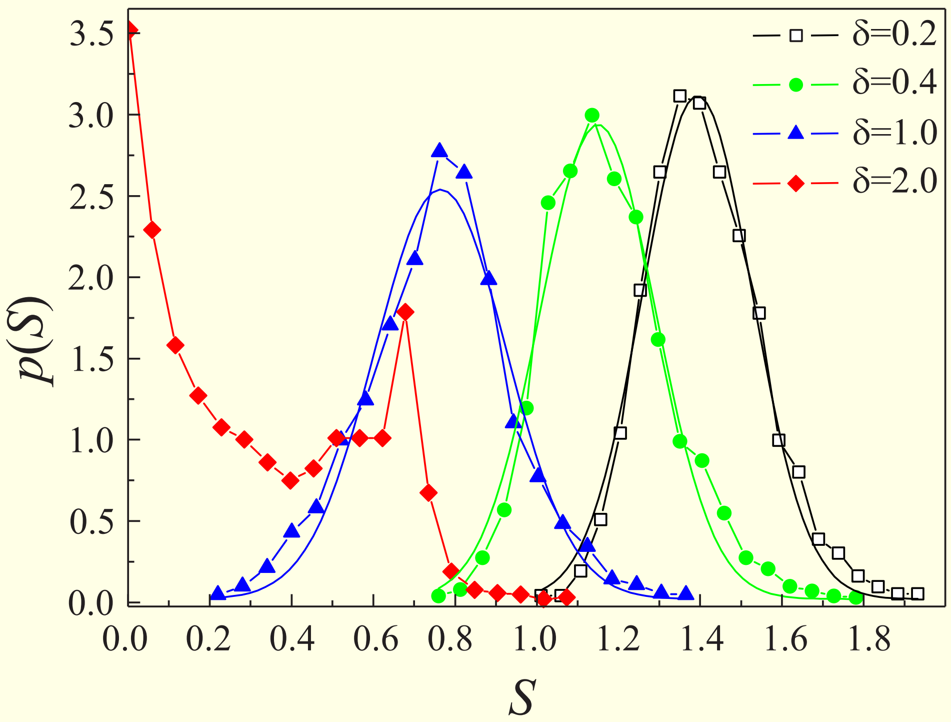
$$\mathbf{Var}\{\varphi(\xi)\} \geq (\mathbf{E}\{\varphi(\xi) - \varphi(\xi + t)\})^2 / F(t).$$

with $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $t > 0$ and a random variable $\xi \geq 0$ satisfying the conditions of the theorem. Choose $\xi = V(0)$ and $\varphi = S_-$ to obtain

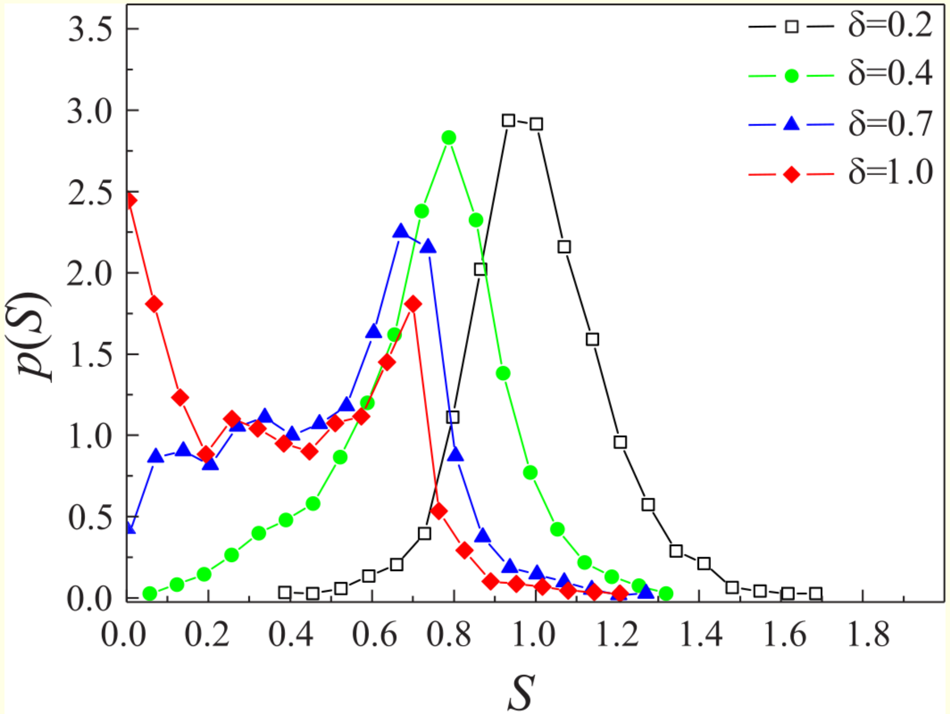
$$\mathbf{Var}\{S_\Lambda\} \geq (\mathbf{E}\{S_-\} - \mathbf{E}\{S_-^t\})^2 / F(t),$$

where $S_-^t = \mathrm{Tr}\ h(P_{\mathbb{Z}_-}^t)$ and P^t is the Fermi projection corresponding to $V(0) + t$. Prove that $\mathbf{E}\{S_-^t\} \to 0,\ t \to \infty$.

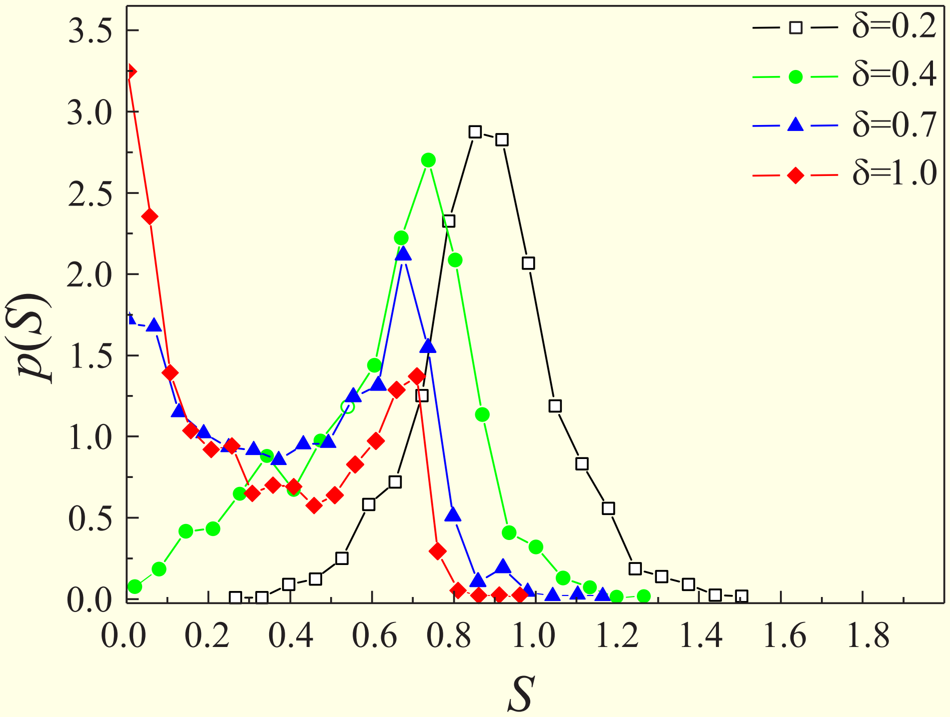
3.10 More Numerics



The probability density of the entanglement entropy $S_{[-M,M]}$ for different values of disorder parameter δ and the uniform distribution of potential.



The probability density of the entanglement entropy $S_{[-M,M]}$ with different disorder parameters δ for the exponential distribution.



The probability density of the entanglement entropy $S_{[-M,M]}$ with different disorder parameters δ for the ”half”-Cauchy distribution.