Homogenization of viscous Hamilton-Jacobi equations with non-convex Hamiltonians: examples and open questions

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Random Event in Honor of Ilya Goldsheid, 12/22/2017

Viscous HJ equation

Consider the following Cauchy problem in $[0,\infty)\times\mathbb{R}^d$

$$\partial_t u^\varepsilon - \varepsilon\,\mathrm{tr}\left(A\left(\frac{x}{\varepsilon}\right)D_x^2 u^\varepsilon\right) + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon\right) = 0, \qquad (\mathrm{HJ}_\varepsilon)$$

$$u^\varepsilon\big|_{t=0} = g(x). \quad \text{(For } \varepsilon = 1 \text{ we write } u \text{ in place of } u^1.\text{)}$$

Here $A(x) = (\sigma\sigma^T)(x)$ is a positive semi-definite matrix:
(A1) $\|\sigma(x)\| \le \Lambda_0$; (A2) $\|\sigma(x)-\sigma(y)\| \le \Lambda_0|x-y|$;
and the Hamiltonian $H(x,p)$ satisfies
(H1) $H \in \mathrm{UC}(\mathbb{R}^d \times B_R)$ for all $R > 0$;
(H2) $\underline{\alpha}(|p|) \le H(x,p) \le \overline{\alpha}(|p|)$ for all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$,
where $\underline{\alpha}(R) \to \infty$ as $R \to \infty$.
We say that (HJ$_\varepsilon$) *homogenizes* if there is a continuous $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ such that for every $g \in \mathrm{UC}(\mathbb{R}^d)$

$$u^\varepsilon(t,x) \rightrightarrows_{\text{loc}} \overline{u}(t,x) \quad \text{as } \varepsilon \to 0,$$

where $\overline{u}$ solves $\quad \partial_t\overline{u} + \overline{H}(D_x\overline{u}) = 0, \quad \overline{u}\big|_{t=0} = g.$

Periodic setting

When $H(x, p)$ is periodic in each of the spatial variables, i.e. $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, homogenization is known to take place under very general conditions.

- (Inviscid case: $A \equiv 0$.) P.-L. Lions, G. Papanicolaou, S.R.S. Varadhan (around 1987, unpublished) suggested two methods:
  - (i) based on construction of correctors;
  - (ii) based on variational representation of solutions (only for H convex in p).
- (Fully non-linear 1-st and 2-nd order equations.) L.C. Evans (1992) used correctors and his perturbed test function method to show homogenization.

Stationary ergodic setting

- *Environment:* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{R}^d$ acts on Ω by shifts $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, which preserve $\mathbb{P}$. We assume that the action by $\tau_x,\ x \in \mathbb{R}^d$, is ergodic.
- *Coefficients:*

$$A(x,\omega) := \tilde{A}(\tau_x\omega), \quad H(x,p,\omega) := \tilde{H}(p,\tau_x\omega).$$

  We shall always assume that A and H satisfy (A1)–(A2) and (H1)–(H2) respectively with bounds independent of ω.
- *Example:* let $A \equiv I$ (viscous) or $A \equiv 0$ (inviscid), and

$$H(x,p,\omega) = \frac{1}{2}|p|^2 - b(x,\omega)|p| + V(x,\omega),$$

  where $b,\ V \in \text{Lip}(\mathbb{R}^d)$, $0 \le b(x,\omega),\ V(x,\omega) \le C$.

Two words about correctors

A function $v_\theta(x,\omega)$ is said to be a (sublinear) corrector corresponding to $\theta \in \mathbb{R}^d$ if a.s. $|x|^{-1} v_\theta(x,\omega) \rightrightarrows_{\text{loc}} 0$ as $|x| \to \infty$ and there is a constant $\overline{H}(\theta) \in \mathbb{R}$ such that

$$-\operatorname{tr}\left(A\left(x,\omega\right) D_x^2 v_\theta\right) + H\left(x, \theta + D_x v_\theta, \omega\right) = \overline{H}(\theta), \quad x \in \mathbb{R}^d.$$

If v_θ is a corrector then $v_\theta^\varepsilon(t,x,\omega) := \theta \cdot x - t\overline{H}(\theta) + \varepsilon v_\theta(x/\varepsilon, \omega)$

- solves (HJ$_\varepsilon$);
- satisfies $v_\theta^\varepsilon(0,x) = \theta \cdot x + \varepsilon v(x/\varepsilon,\omega)$;
- $v_\theta^\varepsilon \rightrightarrows_{\text{loc}} \theta \cdot x - t\overline{H}(\theta)$, a solution of $\partial_t \overline{u} + \overline{H}(D_x \overline{u}) = 0$;

If correctors exist for all $\theta \in \mathbb{R}^d$ then the perturbed test function method gives homogenization for general initial data. However,

- Correctors need not exist in general (stat. erg. setting).[1]
- Yet if (HJ$_\varepsilon$) homogenizes in probability, correctors exist for all θ which are extreme points of sub-level sets of $\overline{H}(\theta)$.[2]

[1] P.-L. Lions, P. E. Souganidis (2003)

[2] P. Cardaliaguet, P. Souganidis (2017+)

Linear initial data characterize $\overline{H}$

$$\partial_t u^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\frac{x}{\varepsilon},\omega\right) D_x^2 u^\varepsilon\right) + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon, \omega\right) = 0. \qquad (\mathrm{HJ}_\varepsilon)$$

If $g(x) = \theta \cdot x$ then we denote the solution of (HJ$_\varepsilon$) by $u_\theta^\varepsilon(t,x,\omega)$.

- $u_\theta^\varepsilon(t,x,\omega) = \varepsilon u_\theta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\right)$, where u_θ solves (HJ$_1$) with $u_\theta\big|_{t=0} = \theta \cdot x$.
- $\overline{u}_\theta(t,x) := \theta \cdot x - t\overline{H}(\theta)$ solves $\partial_t \overline{u}_\theta + \overline{H}(D_x \overline{u}_\theta) = 0$ with $\overline{u}_\theta\big|_{t=0} = \theta \cdot x$.

If (HJ$_\varepsilon$) homogenizes a.s. then, in particular,

$$\forall \theta \in \mathbb{R}^d, \;\; u_\theta^\varepsilon(1,0,\omega) \to -\overline{H}(\theta) \text{ a.s. as } \varepsilon \to 0.$$

Thus, $\overline{H}(\theta)$ is completely characterized by $\lim_{\varepsilon\to 0} u_\theta^\varepsilon(1,0,\omega)$.

- *Question:* Does the above convergence alone imply the full homogenization result?
- *Short answer:* In the stationary ergodic setting and under further conditions on u_θ, yes.

Theorem (A. Davini, EK (2017), stationary ergodic setting)

Assume that

- $\forall \omega \in \Omega$ *the Cauchy problem for (HJ$_1$) is well-posed;*
- *there is a locally bounded function* $\kappa : \mathbb{R}^d \to [0, \infty)$ *such that* $\forall\, x, y \in \mathbb{R}^d,\ \forall t \geq 0,\ \forall \omega \in \Omega$

$$|u_\theta(t, x, \omega) - u_\theta(t, y, \omega)| \leq \kappa(\theta)|x - y| \;;$$

- $\forall \theta \in \mathbb{R}^d$, $u^\varepsilon(1, 0, \omega) \to -\overline{H}(\theta)$ *a.s. as* $\varepsilon \to 0$.

Then $\overline{H}$ *is continuous, coercive, and on a set of full* $\mathbb{P}$*-measure (HJ$_\varepsilon$) homogenizes to* $\partial_t \overline{u} + \overline{H}(D\overline{u}) = 0$.

Well-posedness

- If $A \equiv 0$ then the Cauchy problem for (HJ$_1$) is well-posed (in a certain class) for all H which satisfy (H1)–(H2).
- If $A \not\equiv 0$ then the well-posedness can be shown[3] for A satisfying (A1)–(A2) and $H \in \mathcal{H}(\underline{\alpha}_0, \overline{\alpha}_0, \gamma)$ for some $\underline{\alpha}_0,\ \overline{\alpha}_0 > 0,\ \gamma > 1$. The latter consists of $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$ such that:
  (i) $\underline{\alpha}_0|p|^\gamma - 1/\underline{\alpha}_0 \leq H(x,p) \leq \overline{\alpha}_0(|p|^\gamma + 1) \quad \forall x, p \in \mathbb{R}^d$;
  (ii) $|H(x,p) - H(y,p)| \leq \overline{\alpha}_0(|p|^\gamma + 1)|x - y| \quad \forall x, y, p \in \mathbb{R}^d$;
  (iii) $|H(x,p) - H(x,q)| \leq \overline{\alpha}_0(|p| + |q| + 1)^{\gamma-1}|p - q| \quad \forall x, p, q \in \mathbb{R}^d$.

[3] S. Armstrong, H. Tran (2015), A. Davini (2016+).

Inviscid case: $A \equiv 0$

- Convex H: P. E. Souganidis (1999); F. Rezakhanlou, J. E. Tarver (2000);
- Level set convex H: A. Davini, A. Siconolfi (2009), $d = 1$; S. Armstrong, P. E. Souganidis (2013), $d \geq 1$.
- S. Armstrong, H. V. Tran, Y. Yu (2015): homogenization for $\tilde{H}(p, \omega) = (|p|^2 - 1)^2 - \tilde{V}(\omega)$, $d \geq 1$.
- S. Armstrong, H.V. Tran, Y. Yu (2016), H. Gao (2016): $d = 1$, quite general H.
- B. Ziliotto (2017): counterexample for $d = 2$, $H(p) - V(\omega)$ where $H(p)$ has a strict saddle point and the environment has very slow mixing;
- W. M. Feldman, P. E. Souganidis (2017). Extentions of Ziliotto's example. Homogenization for H with strictly star-shaped sub-level sets.

Viscous case: $A \not\equiv 0$

- Convex H: P.-L. Lions, P. E. Souganidis (2005, 2010); EK, F. Rezakhanlou, S. R. S. Varadhan (2006); S. Armstrong, H. V. Tran (2015).
- $d \geq 1$, $A \equiv I$: S. Armstrong, P. Cardaliaguet (2015+). H is such that
  $\exists \alpha > 0: \ \tilde{H}(te, \omega) = t^\alpha \tilde{H}(e, \omega) \ \forall t \geq 0, \ \|e\| = 1, \ x \in \mathbb{R}^d$.
  Environments with finite range of dependence.
- $d = 1$, A is general: A. Davini, EK (2017). "Pinned Hamiltonians": $H(\cdot, p_0) \equiv const$ for some p_0. Examples include: $\tilde{H}(p, \omega) = \tilde{a}(\omega)|p|^\alpha - \tilde{b}(\omega)|p|$, $\alpha > 1$,
  $0 < c \leq \tilde{a}(\omega), \tilde{b}(\omega) \leq c^{-1}$.
- $d = 1$, $A \equiv 1$: A. Yilmaz, O. Zeitouni (2017+); EK, A. Yilmaz, O. Zeitouni (2017+). $\tilde{H}(p, \omega) = \frac{1}{2} p^2 - c|p| + \beta V(\omega)$, $\beta, c > 0$, under additional conditions on the environment.

“Pinned Hamiltonians”

Let $d = 1$, A satisfy (A1)–(A2), and $H_\pm \in \mathcal{H}(\underline{\alpha}_0, \overline{\alpha}_0, \gamma)$, where bounds and parameters are independent of ω. Assume, in addition, that (HJ$_\varepsilon$) with Hamiltonians $\tilde{H}_\pm$ homogenizes, $\tilde{H}_\pm(0,\omega) \equiv 0$, and

$$\tilde{H}_+(p,\omega)p \le \tilde{H}_-(p,\omega)p \quad \forall p \in \mathbb{R},\ \omega \in \Omega.$$

Define

$$\tilde{H}(p,\omega) := \min\{\tilde{H}_+(p,\omega), \tilde{H}_-(p,\omega)\} = \begin{cases} \tilde{H}_+(p,\omega), & \text{if } p \ge 0; \\ \tilde{H}_-(p,\omega), & \text{if } p \le 0. \end{cases}$$

Example

The homogenization requirement for $\tilde{H}_\pm$ is met if, for example,

- $\tilde{H}_\pm$ are convex in p;
- $A \equiv 0$ and $\tilde{H}_\pm$ are level set convex;
- $\tilde{H}_\pm$ (up to a shift by a fixed p_0 and adding a constant) are of the form for which we already obtained homogenization.

Recall $H(x,p,\omega) = \tilde{H}(p,\tau_x\omega)$, where

$$\tilde{H}(p,\omega) = \begin{cases} \tilde{H}_+(p,\omega), & \text{if } p \geq 0; \\ \tilde{H}_-(p,\omega), & \text{if } p \leq 0. \end{cases} \quad \text{and } \tilde{H}_\pm(0,\omega) \equiv 0.$$

Theorem (A. Davini, EK (2017))

Let $d = 1$, A and H be as above. Then there exist a continuous and coercive Hamiltonian $\overline{H} : \mathbb{R} \to \mathbb{R}$ and a set $\hat{\Omega} \subseteq \Omega$ of full measure such that the equation

$$\partial_t u^\varepsilon - \varepsilon A\left(\frac{x}{\varepsilon},\omega\right) D_x^2 u^\varepsilon + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon, \omega\right) = 0$$

homogenizes. Moreover,

$$\overline{H}(\theta) = \min\{\overline{H}_-(\theta), \overline{H}_+(\theta)\} \qquad \forall \theta \in \mathbb{R},$$

where $\overline{H}_+$ and $\overline{H}_-$ are the effective Hamiltonians obtained by homogenizing (HJ$_\varepsilon$) with H_+ and H_- in place of H.

$H(p,x,\omega) = \frac{1}{2}p^2 - c|p| + \beta V(x,\omega),\ \beta, c > 0$

Let ess sup $V(x,\omega) = 1$ and ess inf $V(x,\omega) = 0$.

- *Valleys and hills:* for $\omega \in \Omega$ and $h \in (0,1)$, an interval I is an h-valley (h-hill) if $\forall x \in I,\ V(x,\omega) \le h$ ($V(x,\omega) \ge h$).
- *Assumption:* $\forall h \in (0,1)$ and $y > 0$,
  $\mathbb{P}([0,y]$ is an h-valley$) > 0$ and $\mathbb{P}([0,y]$ is an h-hill$) > 0$.

Consider the equation

$$\partial_t u_\theta - \frac{1}{2} D_x^2 u_\theta + \frac{1}{2} (D_x u_\theta)^2 - c|D_x u_\theta| + \beta V(x,\omega) = 0$$

with the initial condition $u_\theta(0,x,\omega) = \theta x$. We are interested in the limit of $u_\theta^\varepsilon(1,0,\omega) := \varepsilon u_\theta(\frac{1}{\varepsilon},0,\omega)$ as $\varepsilon \to 0$.

Solution by Hopf-Cole + control representation

Note that $v_\theta := e^{-u_\theta}$ solves

$$\partial_t v_\theta - \frac{1}{2} D_x^2 v_\theta \underbrace{+c\,|D_x v_\theta|}_{-\inf_{|b|\le c}(bD_x v_\theta)} -\beta V(x,\omega) v_\theta = 0, \quad v_\theta\big|_{t=0} = e^{-\theta x}.$$

The control representation gives

$$u_\theta(t,x,\omega) = -\ln v_\theta(t,x,\omega) = -\ln \inf_{b\in\mathcal{P}_c} E_x e^{-\theta x(t) + \beta\int_0^t V(x(s),\omega)\,ds},$$

where $dx(s) = b(s)\,ds + dB(s),\ 0 \le s \le t$, $x(0) = x$, and

$\mathcal{P}_c = \{b = (b(s))_{s\ge 0} :\ b$ is a $[-c,c]$-valued and progress. meas.$\}$.

Thus we are interested in the limit as $T = \varepsilon^{-1} \to \infty$ of

$$\inf_{b\in\mathcal{P}_c} \frac{1}{T} \ln E_0 e^{-\theta x(T) + \beta\int_0^T V(x(s),\omega)\,ds}.$$

Recall that $\overline{H}(\theta) = -\lim_{\varepsilon\to 0} u^\varepsilon(1,0,\omega)$ (if it exists) is equal to

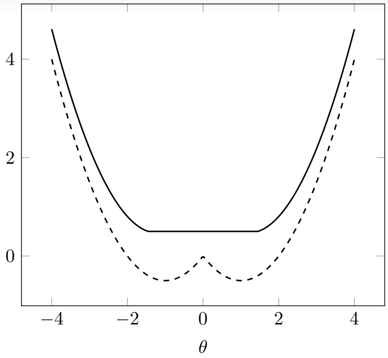
$$\lim_{T\to\infty} \inf_{b\in\mathcal{P}_c} \frac{1}{T} \ln E_0 e^{-\theta x(T)+\beta\int_0^T V(x(s),\omega)\,ds}.$$

When $c = 0$, the limit exists and is sometimes referred to as the tilted free energy of a BM in the potential V,

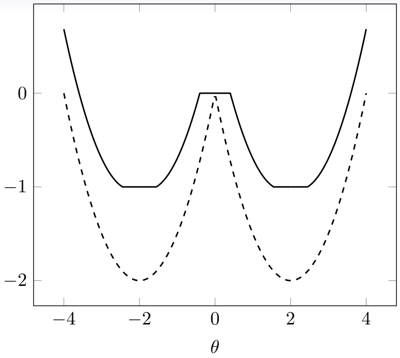
$$\Lambda_\beta(\theta) := \lim_{T\to\infty} \frac{1}{T} \ln E_0 e^{\theta B(T)+\beta\int_0^T V(B(s),\omega)\,ds}.$$

Using a variety of techniques, including construction of correctors, asymptotically optimal policies, large deviations, we prove homogenization and give an explicit formula for $\overline{H}(\theta)$ in terms of $\Lambda_\beta(\theta)$.

Weak control $[\beta = 1,\ c = 1]$



Strong control $[\beta = 1,\ c = 2]$



Weak control: $\beta \geq \frac{c^2}{2}$

$$\overline{H}(\theta) = \begin{cases} \beta - \frac{c^2}{2}, & \text{if } |\theta| \leq c; \\ \Lambda_\beta(|\theta| - c) - \frac{c^2}{2}, & \text{if } |\theta| > c. \end{cases}$$

Pictures by A. Yilmaz

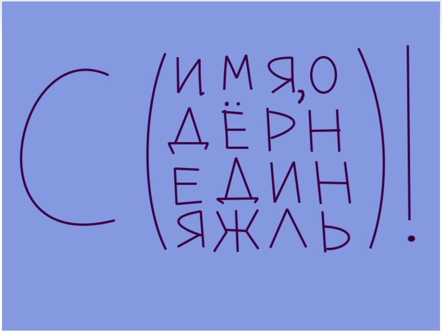
Strong control: $\beta < \frac{c^2}{2}$

$$\overline{H}(\theta) = \begin{cases} 0, & \text{if } |\theta| \leq |\overline{c}|; \\ \Lambda(|\theta| - c) - c^2/2, & \text{if } |\theta| > |\overline{c}|, \end{cases}$$

where $\overline{c} \in (0, c)$ is a unique solution of the equation $\Lambda_\beta(\overline{c} - c) = \frac{c^2}{2}$.

Some open problems

- Prove homogenization under some natural assumptions for $A = I$ and $H(x, p, \omega) = \frac{1}{2}|p|^2 - b(x, \omega)|p| + V(x, \omega)$, $d \geq 1$.
- Suppose that $H(x, p, \omega) = H_0(p) - \varepsilon|p| + V(x, \omega)$ where $H_0(p)$ is a convex super-linear Hamiltonian, $\varepsilon > 0$, and V is bounded and sufficiently regular. Is it true that (HJ$_\varepsilon$) homogenizes for all sufficiently small $\varepsilon > 0$?
- Study the "level-set-convexification of the effective Hamiltonian". For partial results and a conjecture in the inviscid case see J. Qian, H. V. Tran, Y. Yu (2017+).
- Prove homogenization for level-set convex Hamiltonians in the viscous case.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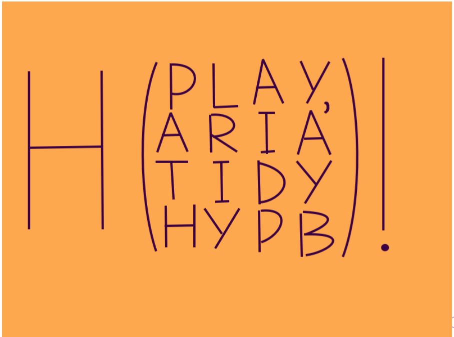
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