Limit Theorems for Nonconventional Arrays

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For almost 10 years now I'm studying various limit theorems for nonconventional sums

$$S_N = \sum_{n=1}^{N} F(X(n), X(2n), ..., X(\ell n))$$

and more general ones, usually, under weak dependence conditions on a stochastic sequence $X(n)$, $n = 1, 2, ...$ which, in particular, can be generated by a dynamical system $X(n) = T^n f = f \circ T^n$ on a probability space $(\mathcal{X}, \mathcal{F}, P)$ with P-preserving T. Important: Summands in S_N are strongly dependent, for instance, $F(X(n), X(2n))$ and $F(X(2n), X(4n))$.
Part of the motivation came from nonconventional ergodic theorems and their connection with the Szemerédi theorem on arithmetic progressions in sets of integers of positive density where the sums

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were considered.

a) Number theory (combinatorial) applications

For each $\omega \in [0,1)$ consider its base m or continued fraction expansions with digits $\xi_k(\omega)$, $k = 1, 2, \ldots$. Count the number of those $n \leq N$ for which, say, $\xi_{jn}(\omega) = a_j$, $j = 1, \ldots, \ell$ for some fixed integers $a_1, \ldots, a_\ell$. Then

$$S_N(\omega) = \sum_{n=1}^{N} \prod_{j=1}^{\ell} \delta_{a_j \xi_{jn}(\omega)}$$

where $\delta_{km} = 1$ if $k = m$ and $= 0$, otherwise.

b) Arithmetic progressions in a random set

Define a random set Γ in positive integers via a sequence of random variables $\xi_1, \xi_2, \ldots$ taking on values 0 or 1 by saying that $n \in \Gamma$ iff $\xi_n = 1$. Then

$$S_N = \sum_{n=1}^{N} \prod_{j=1}^{\ell} \xi_{jn}$$

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More recently I thought that it makes sense to consider more general sums

$$S_N = \sum_{n=1}^{N} F(X(n), X(2n), ..., X(\ell n), X(N-n), X(2(N-n)), ..., X(\ell(N-n)))$$

which look natural and symmetric. In fact, it turned out natural and convenient to consider more general situation

$$S_N = \sum_{n=1}^{N} F(X(p_1 n + q_1 N), ..., X(p_\ell n + q_\ell N))$$

where all p_j's are different and ordered so that $p_1 < p_2 < ... < p_\ell$. In particular, if $\ell = 2k$, $p_j = -(k-j+1)$, $q_j = (k-j+1)$ for $j = 1, ..., k$ and $p_j = j-k$, $q_j = 0$ for $j = k+1, ..., 2k$ we arrive at the previous case.
Summands depend on N: Nonconventional (triangular) arrays.
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Next, we will discuss almost sure (pointwise) convergence of averages of nonconventional arrays

$$N^{-1}S_N = N^{-1}\sum_{n=1}^{N} F(X(p_1 n + q_1 N), X(p_2 n + q_2 N), ..., X(p_\ell n + q_\ell N))$$

for a function $F = F(x_1, ..., x_\ell)$ satisfying certain conditions and for a sequence of weakly dependent random variables $X(1)$, $X(2)$, ... which could be generated also by a dynamical system $X(n) = T^n f$.

The setup below will work for several applications though it is not strongest possible to avoid too many technicalities. We assume that random variables $X(1)$, $X(2)$, ... are defined on a probability space $(\Omega, \mathcal{F}, P)$ where we have also a family of σ-algebras $\mathcal{F}_{kl} \subset \mathcal{F}$, $0 \leq k \leq l \leq \infty$ such that $\mathcal{F}_{kl} \subset \mathcal{F}_{k'l'}$ if $k' \leq k$ and $l' \geq l$. We introduce also the ϕ-dependence coefficient by

$$\phi(n) = \sup_{k,A,B}\{|\frac{P(A\cap B)}{P(A)} - P(B)| : A \in \mathcal{F}_{-\infty,k},\ B \in \mathcal{F}_{k+n,\infty},\ P(A) > 0\}.$$

If $\phi(n) \to 0$ when $n \to \infty$ then the above family of σ-fields are called ϕ-mixing.

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If $\phi(n) \to 0$ when $n \to \infty$ then the above family of σ-fields are called ϕ-mixing.

We will need also the approximation coefficient

$$\beta(n) = \sup_{k\geq 1} \|X(k) - E(X(k)|\mathcal{F}_{k-n,k+n})\|_{L_\infty}$$

and assume that $F = F(x_1, ..., x_\ell)$ is a bounded Hölder continuous function i.e.,

$$|F(x)| \leq M \text{ and } |F(x) - F(y)| \leq M \sum_{i=1}^{\ell} |x_i - y_i|^\kappa, \ \forall\ x = (x_1, ..., x_\ell),\ y = (y_1, ..., y_\ell).$$

Theorem

Suppose that all $X(n)$'s have the same distribution μ,

$$\sum_{n=1}^{\infty} (\phi(n) + \beta^\kappa(n)) < \infty$$

and all p_j's are distinct. Then almost surely

$$\lim_{N\to\infty} \frac{1}{N} S_N = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(X(p_1 n + q_1 N), X(p_2 n + q_2 N), ..., X(p_\ell n + q_\ell N)) = \bar{F} = \int F(x_1, x_2, ..., x_\ell) d\mu(x_1) d\mu(x_2) ... d\mu(x_\ell).$$

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Our mixing conditions introduced via dependence coefficients of a two parameter family of σ-algebras $\mathcal{F}_{kl}$ are suited both for applications to discrete time stochastic processes and to dynamical systems, the latter having appropriate symbolic representations. If, say ξ_k, $k \geq 0$ is a Markov chain we can take σ-algebras $\mathcal{F}_{kl}$, $k \leq l$ generated by $\xi_k, \xi_{k+1}, ..., \xi_l$ and $X(n) = f(\xi_n, \xi_{n+1}, ...)$ with bounded functions f on the sequence space sufficiently weakly dependent on tails so that the assumption on the approximation coefficient $\beta(n)$ will be satisfied. Then, for instance, under the Doeblin condition our mixing assumptions will be met.

On the dynamical systems side we can take a subshift of finite type T acting on a space of sequences with a finite alphabet where the (finite) σ-algebra $\mathcal{F}_{kl}$ will be generated by cylinder sets with fixed coordinates on places from k to l. Under the topologically mixing assumption Gibbs shift invariant measures constructed by Hölder continuous functions will meet our mixing assumptions and we can set $X(n) = f \circ T^n$ for some Hölder continuous function on the sequence space. The results remain true for such dynamical systems as Axiom A diffeomorphisms and expanding transformations which have symbolic representations as subshifts of finite type via their Markov partitions. The results are also applicable to some dynamical systems which have symbolic representations with infinite alphabets such as the Gauss map $Tx = \frac{1}{x} \pmod 1$ which is exponentially fast ϕ-mixing (even ψ-mixing) with respect to its Gauss invariant measure $G(\Gamma) = \frac{1}{\ln 2} \int_\Gamma \frac{dx}{1+x}$.

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Without loss of generality assume $\bar{F} = \int F(x_1, x_2, ..., x_\ell) d\mu(x_1) d\mu(x_2) ... d\mu(x_\ell) = 0$. Our goal is to show that $ES_N^4 \leq CN^2$ and then use Borel-Cantelli lemma to have that $\frac{1}{N} S_N \to 0$ with probability one. Even estimating ES_N^2 where we have to study covariances

$$E\big(F(X(p_1 n + q_1 N), X(p_2 n + q_2 N), ..., X(p_\ell n + q_\ell N))$$
$$\times F(X(p_1 m + q_1 N), X(p_2 m + q_2 N), ..., X(p_\ell m + q_\ell N))\big)$$

the situation is not simple even when $X(n)$'s are i.i.d. because indexes can coincide (or be close).

To make things computable we split $\{1, 2, ..., N\}$ into no more than $\ell!$ subintervals $\mathcal{N}_{\varepsilon,N} = \{n : a_\varepsilon N < n < b_\varepsilon N\}$ and a set $\tilde{\mathcal{N}}_N$ of cardinality not exceeding ℓ^2 so that if $n \in \mathcal{N}_{\varepsilon,N}$, $\varepsilon = (\varepsilon_1, ..., \varepsilon_\ell))$ then

$$p_{\varepsilon_i(n,N)} n + q_{\varepsilon_i(n,N)} N < p_{\varepsilon_{i+1}(n,N)} n + q_{\varepsilon_{i+1}(n,N)} N \text{ for all } i = 1, 2, ..., \ell - 1.$$

The following lemma allows an effective use of this splitting with the goal to make estimates of moments of sums along each interval $\mathcal{N}_{\varepsilon,N}$ separately.

Lemma

If $a_\varepsilon N < n < b_\varepsilon N$ then

$$(p_{\varepsilon_{i+1}} - p_{\varepsilon_i})n + (q_{\varepsilon_{i+1}} - q_{\varepsilon_i})N \geq \min(n - a_\varepsilon N - 1,\ b_\varepsilon N - n - 1).$$

Ideas of the proof

Without loss of generality assume
$\bar{F} = \int F(x_1, x_2, ..., x_\ell) d\mu(x_1) d\mu(x_2) ... d\mu(x_\ell) = 0$. Our goal is to show that $ES_N^4 \le CN^2$ and then use Borel-Cantelli lemma to have that $\frac{1}{N} S_N \to 0$ with probability one. Even estimating ES_N^2 where we have to study covariances

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Splitting of functions

For each $\varepsilon=(\varepsilon_1,...,\varepsilon_\ell)\mathcal{E}_\ell$ we define

$$F_{\ell,\varepsilon}(x_{\varepsilon_1},...,x_{\varepsilon_\ell})=F(x_1,...,x_\ell)-\int F(x_1,...,x_\ell)d\mu(x_{\varepsilon_\ell})\text{ and}$$

$$F_{j,\varepsilon}(x_{\varepsilon_1},...,x_{\varepsilon_j})=\int F(x_1,...,x_\ell)d\mu(x_{\varepsilon_\ell})d\mu(x_{\varepsilon_{\ell-1}})\cdots d\mu(x_{\varepsilon_{j+1}})$$
$$-\int F(x_1,...,x_\ell)d\mu(x_{\varepsilon_\ell})\cdots d\mu(x_{\varepsilon_j}).$$

Observe that $EF_{j,\varepsilon}(x_{\varepsilon_1},...,x_{\varepsilon_{j-1}},\xi(n))=0$. For $j=1,...,\ell$ set

$$S_{j,\varepsilon}(N)=\sum_{n\in\mathcal{N}_{\varepsilon,N}}F_{j,\varepsilon}\big(\xi(p_{\varepsilon_1}n+q_{\varepsilon_1}N),\xi(p_{\varepsilon_2}n+q_{\varepsilon_2}N),...,\xi(p_{\varepsilon_j}n+q_{\varepsilon_j}N)\big).$$

Then

$$S_N=\sum_{\varepsilon\in\mathcal{E}_\ell}\sum_{j=1}^{\ell}S_{j,\varepsilon}(N)+\sum_{n\in\hat{\mathcal{N}}_N}F(\xi(p_1n+q_1N),...,\xi(p_\ell n+q_\ell N)).$$

It "remains" to estimate "easier" covariances of the form

$$E\big(F_{i,\varepsilon}(\xi(p_{\varepsilon_1}m+q_{\varepsilon_1}N),\xi(p_{\varepsilon_2}m+q_{\varepsilon_2}N),...,\xi(p_{\varepsilon_i}m+q_{\varepsilon_i}N))$$
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The next natural question to study is whether $N^{-1/2}(S_N - \bar{F}) = N^{-1/2}\sum_{n=1}^{N}(F(X(p_1 n + q_1 N), X(p_2 n + q_2 N), ..., X(p_\ell n + q_\ell N)) - \bar{F})$ converges weakly as $N \to \infty$ to a normal distribution. For $q_i = 0, \forall i$ this is true (K-Varadhan). It is necessary for such convergence that the variance $N^{-1}Var(S_N) = N^{-1}E(S_N - \bar{F})^2$ converges as $N \to \infty$. It turns out that, in general, this is not true for the above expressions.

Example Let $X(n), n \geq 0$ be i.i.d. with a distribution μ and $S_N = \sum_{n=1}^{N} F(X(2n+N), X(2N-2n))$ with $F(x,y) = F(y,x)$, $\int F(x,y)d\mu(x)d\mu(y) = 0$ then

$$\lim_{N\to\infty, N\,\text{odd}} \frac{1}{N}ES_N^2 = \int F^2(x,y)d\mu(x)d\mu(y) \neq \lim_{N\to\infty, N\,\text{even}} \frac{1}{N}ES_N^2$$
$$= 2\int F^2(x,y)d\mu(x)d\mu(y).$$

Still, we have the following result (which includes $S_N = \sum_{n=1}^{N}(F(X(k(N-n)), ..., X(2(N-n)), X(N-n), X(n), X(2n), ..., X(kn)))$.

Theorem

Let all $X(n)$ have the same distribution and $(X(m), X(n)) \sim (X(0), X(n-m))$. Assume that each difference $q_i - q_j$ is divisible by the greatest common divisor of p_i and p_j, $i, j = 1, 2, ..., \ell$ and, as before, that $\sum_{n=1}^{\infty}(\phi(n) + \beta^{\kappa}(n)) < \infty$. Then $\lim_{N\to\infty} N^{-1}Var(S_N) = \sigma^2$ exists and $N^{-1/2}(S_N - \bar{F})$ converges in distribution as $N \to \infty$ to the normal random variable with mean 0 and variance σ^2.

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Idea of the proof

In the proof we use the splitting/decomposition described earlier, construct martingale approximations of the sums S_N and prove convergence of covariances. Namely, we study the asymptotical behavior as $N \to \infty$ of covariances

$$D_{i,j,\varepsilon,\tilde{\varepsilon}}(N) = \frac{1}{N} E S_{i,\varepsilon}(N) S_{j,\tilde{\varepsilon}}(N) = \frac{1}{N} \sum_{m \in \mathcal{N}_{\varepsilon,N}, n \in \mathcal{N}_{\tilde{\varepsilon},N}} b_{i,j,\varepsilon,\tilde{\varepsilon}}(N,m,n)$$

$b_{i,j,\varepsilon,\tilde{\varepsilon}}(N,m,n) = E\big(Y_{i,\varepsilon,\rho_{\varepsilon_i}(m,n)} Y_{j,\tilde{\varepsilon},\rho_{\tilde{\varepsilon}_j}(m,n)}\big)$, $\rho_{\varepsilon_j}(n,N) = p_{\varepsilon_j} n + q_{\varepsilon_j} N$,
$Y_{j,\varepsilon,\rho_{\varepsilon_j}(n,N)} = F_{j,\varepsilon}(\xi(\rho_{\varepsilon_1}(n,N)),...,\xi(\rho_{\varepsilon_j}(n,N))$ and $Y_{j,\varepsilon,m} = 0$ if $m \neq \rho_{\varepsilon_j}(n,N)$.
We compute limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{a_\varepsilon N < m < b_\varepsilon N, a_{\tilde{\varepsilon}} N < n < b_{\tilde{\varepsilon}} N} b_{i,j,\varepsilon,\tilde{\varepsilon}}(N,m,n).$$

Actually, we show first that if $N \to \infty$,
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in $m \in \mathcal{N}_{\varepsilon,N}$ and $n \in \mathcal{N}_{\tilde{\varepsilon},N}$, divide this number by N and show that this ratio converges to a limit which will give the limit we want. Finally, we just sum in u.

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Classical Poisson theorem can be formulated in the following way: If $\xi_1, \xi_2, \dots$ are i.i.d. random variables, A_n is a sequence of sets such that $\mathbb{P}\{\xi_1 \in A_n\} \downarrow 0$ and $N \sim \lambda(\mathbb{P}\{\xi_1 \in A_N\})^{-1}$, $\lambda > 0$ then $S_N = \sum_{k=1}^N \mathbb{I}_{A_N}(\xi_k)$ ($\mathbb{I}_A$-indicator of A) converges in distribution to a Poisson random variable with the parameter λ. It turns out that for nonconventional arrays (where p_j's are distinct) we obtain that when $N \sim \lambda(\mathbb{P}\{\xi_1 \in A_N\})^{-\ell}$ then

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may not converge in distribution as $n \to \infty$ to a Poisson random variable with the parameter λ though it was true when all q_j's vanish. Consider the example

$$S_N = \sum_{n=1}^N \mathbb{I}_{\Gamma_N}(\xi(n))\mathbb{I}_{\Gamma_N}(\xi(N-n))$$
$$= 2\sum_{n=1}^{[N/2]} \mathbb{I}_{\Gamma_N}(\xi(n))\mathbb{I}_{\Gamma_N}(\xi(N-n)) + \mathbb{I}_{\Gamma_N}(\xi(0))\mathbb{I}_{\Gamma_N}(\xi(N)) - a_N$$

where $a_N = \mathbb{I}_{\Gamma_N}(\xi([N/2]))$ if N is even and $a_N = 0$ if N is odd. If $\xi(n)$'s have the same distribution μ and $\mu(\Gamma_N) \to 0$ as $N \to \infty$ then both $\mathbb{I}_{\Gamma_N}(\xi(0))\mathbb{I}_{\Gamma_N}(\xi(N))$ and a_N tend to 0 in probability as $N \to \infty$. Thus, any limit in distribution of S_N as $N \to \infty$ could only be a random variable taking on even values, and so it cannot have a Poisson distribution.

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Assumption

For any nontrivial permutation ζ of ℓ numbers $(1, 2, ..., \ell)$ the matrix $\begin{pmatrix} p_1 & p_2 & \cdots & p_\ell \\ p_{\zeta(1)} & p_{\zeta(2)} & \cdots & p_{\zeta(\ell)} \end{pmatrix}$ has rank 2.

Observe that this assumption is satisfied, for instance, when $p_i > 0$ for all $i = 1, ..., \ell$. Indeed, assume as before without loss of generality that $0 < p_1 < p_2 < ... < p_\ell$. If the above matrix has rank 1 for some permutation ζ then $p_{\zeta(i)} = ap_i$, $i = 1, ..., \ell$ for some $a > 0$. But then $p_{\zeta(1)} < p_{\zeta(2)} < ... < p_{\zeta(\ell)}$, and so $p_{\zeta(i)} = p_i$, $i = 1, ..., \ell$, i.e. ζ is a trivial permutation.

Next, we consider the case where $\xi(n)$, $n \geq 0$ is a stationary sequence of random variables. Define σ-algebras $\mathcal{F}_{mn} = \sigma\{\xi(m), \xi(m+1), ..., \xi(n)\}$, $m \leq n$, which is the minimal σ-algebra such that $\xi(m), \xi(m+1), ..., \xi(n)$ are all measurable with respect to it. Then the ψ-dependence (or mixing) coefficient is defined by

$$\psi(n) = \sup_{m \geq 0} \{ | \frac{P(A \cap B)}{P(A)P(B)} - 1 | : A \in \mathcal{F}_{0,m}, \, B \in \mathcal{F}_{m+n,\infty}, \, P(A)P(B) \neq 0 \}$$

where $\mathcal{F}_{k,\infty}$ is the minimal σ-algebra containing all $\mathcal{F}_{k,n}$, $n \geq k$. The sequence $\xi(n)$, $n \geq 0$ is called ψ-mixing if

$$\psi(1) < \infty \quad \text{and} \quad \lim_{n \to \infty} \psi(n) = 0.$$

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Theorem

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We set

$$\eta_n^{(N)} = \prod_{j=1}^{\ell} \mathbb{I}_{\Gamma_N}(\xi(p_j n + q_j N)), \quad n = 1, 2, ..., N$$

and check conditions of the following old criterion of Sevast'yanov.

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Sevast'yanov's theorem

Theorem

Let $\eta_1^{(N)},...,\eta_N^{(N)}$, $N=1,2,...$ *be an array of 0–1 random variables,* $\mathcal{J}_r(N)$, $r\leq N$ *be the family of all* r*-tuples* $(i_1,i_2,...,i_r)$ *of mutually distinct indices between 1 and* N *and for any* $(i_1,...,i_r)\in\mathcal{J}_r(N)$ *set* $b^{(N)}_{i_1,...,i_r}=P\{\eta_{i_1}^{(N)}=...=\eta_{i_r}^{(N)}=1\}$. *Assume that*

$$\lim_{N\to\infty}\max_{1\leq i\leq N} b_i^{(N)}=0,\;\;\lim_{N\to\infty}\sum_{i=1}^N b_i^{(N)}=\lambda>0,$$

for $N=1,2,...$ *there exist "rare" sets* $I_r(N)\subset\mathcal{J}_r(N)$ *such that*

$$\lim_{N\to\infty}\sum_{(i_1,...,i_r)\in I_r(N)} b^{(N)}_{i_1...i_r}=\lim_{N\to\infty}\sum_{(i_1,...,i_r)\in I_r(N)} b^{(N)}_{i_1}\cdots b^{(N)}_{i_r}=0$$

and uniformly in $(i_1,...,i_r)\in\mathcal{J}_r(N)\setminus I_r(N)$,

$$\lim_{N\to\infty} b^{(N)}_{i_1...i_r}(b^{(N)}_{i_1}\cdots b^{(N)}_{i_r})^{-1}=1.$$

$$\text{Then}\;\;\lim_{N\to\infty}P\{\sum_{i=1}^N\eta_i^{(N)}=k\}=\frac{\lambda^k e^{-\lambda}}{k!},\;k=0,1,2,....$$

Theorem

Let $\eta_1^{(N)}, ..., \eta_N^{(N)}$, $N = 1, 2, ...$ *be an array of 0–1 random variables,* $\mathcal{J}_r(N)$, $r \leq N$ *be the family of all* r*-tuples* $(i_1, i_2, ..., i_r)$ *of mutually distinct indices between 1 and* N *and for any* $(i_1, ..., i_r) \in \mathcal{J}_r(N)$ *set* $b^{(N)}_{i_1,\dots,i_r} = P\{\eta^{(N)}_{i_1} = ... = \eta^{(N)}_{i_r} = 1\}$. *Assume that*

$$\lim_{N\to\infty} \max_{1\leq i\leq N} b_i^{(N)} = 0, \quad \lim_{N\to\infty} \sum_{i=1}^{N} b_i^{(N)} = \lambda > 0,$$

for $N = 1, 2, ...$ *there exist "rare" sets* $I_r(N) \subset \mathcal{J}_r(N)$ *such that*

$$\lim_{N\to\infty} \sum_{(i_1,\dots,i_r)\in I_r(N)} b^{(N)}_{i_1\dots i_r} = \lim_{N\to\infty} \sum_{(i_1,\dots,i_r)\in I_r(N)} b^{(N)}_{i_1} \cdots b^{(N)}_{i_r} = 0$$

and uniformly in $(i_1, ..., i_r) \in \mathcal{J}_r(N) \setminus I_r(N)$,

$$\lim_{N\to\infty} b^{(N)}_{i_1\dots i_r} (b^{(N)}_{i_1} \cdots b^{(N)}_{i_r})^{-1} = 1.$$

Then $\lim_{N\to\infty} P\{\sum_{i=1}^{N} \eta_i^{(N)} = k\} = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, ...$

Theorem

Let $\eta_1^{(N)},...,\eta_N^{(N)}$, $N=1,2,...$ *be an array of 0–1 random variables,* $\mathcal{J}_r(N)$, $r\le N$ *be the family of all r-tuples* $(i_1,i_2,...,i_r)$ *of mutually distinct indices between 1 and* N *and for any* $(i_1,...,i_r)\in\mathcal{J}_r(N)$ *set* $b^{(N)}_{i_1,\dots,i_r}=P\{\eta^{(N)}_{i_1}=...=\eta^{(N)}_{i_r}=1\}$. *Assume that*

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$$\text{Then }\lim_{N\to\infty}P\{\sum_{i=1}^N\eta_i^{(N)}=k\}=\frac{\lambda^k e^{-\lambda}}{k!},\;k=0,1,2,....$$

For any two positive integers $l, \tilde{l}$ set

$$d(l,\tilde{l}) = \min_{1\le i,j\le \ell} |p_i l - p_j \tilde{l} + (q_i - q_j)N| \ \text{ and } \ a(N) = |\ln \mu(\Gamma_N)|.$$

A sequence $J = \{j_1, j_2, ..., j_l\}$ of distinct positive integers from $\mathcal{N}_N$ will be called an N-cluster here if for any $j, \tilde{j} \in J$ there exists a chain $j_{i_1}, j_{i_2}, ..., j_{i_{m-1}}, j_{i_m} = \tilde{j}$ of integers from J such that

$$d(j_{i_k}, j_{i_{k+1}}) \le a(N) \quad \forall k = 1, 2, ..., m-1.$$

We say that J is a maximal N-cluster in another finite sequence $\tilde{J}$ of distinct positive integers if $J \cup \{\tilde{j}\}$ is already not an N-cluster for any $\tilde{j} \in \tilde{J} \setminus J$. Let $\mathcal{J}_r(N)$ be the set of all r-tuples of mutually distinct indices between 1 and N. Define rare sets $I_r(N)$ as collections of r-tuples $J = (i_1, i_2, ..., i_r)$ which either contains a cluster with more than one element or

$$i_{\min}(J) = \min_{\varepsilon\in\mathcal{E}_\ell} \min_{1\le l\le r,\, a_\varepsilon N < i_l < b_\varepsilon N} (i_l - a_\varepsilon N, b_\varepsilon N - i_l) \le a(N).$$

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Next, we count multiple arrivals to appropriately shrinking cylinders A_n making left shifts on a sequence space $\Omega = \{\omega = (\omega_i),\ i \geq 0 : \omega_i \in \mathcal{A}\}$ with a finite or countable alphabet $\mathcal{A}$ considering
$A_n(a) = \{\omega = (\omega_i) \in \Omega : \omega_j = a_j,\ j = 0, 1, ..., n-1\}$ where $a = (a_i) \in \Omega$ is a fixed sequence and $(T\omega)_i = \omega_{i+1}$.
Conditioned to the Assumption we prove that

$$S_N^{A_n(a)}(\omega) = \sum_{k=1}^{N} \mathbb{I}_{A_n(a)}(T^{p_1 k + q_1 N}\omega)\mathbb{I}_{A_n(a)}(T^{p_2 k + q_2 N}\omega) \cdots \mathbb{I}_{A_n(a)}(T^{p_\ell k + q_\ell N}\omega)$$

will have asymptotically Poisson distribution for
$N = N_n \sim \lambda(\mathbb{P}(A_n(a)))^{-\ell} \to \infty$ as $n \to \infty$ provided P is ψ-mixing and that the cylinder $A_n(a)$ of length n is built by a nonperiodic sequence a. For cylinders built on periodic points the limit may not exist at all or it could be a compound Poisson distribution.

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