The bulk-edge correspondence for disordered chiral chains

Gian Michele Graf
ETH Zurich

Classical and quantum motion in disordered environment
A random event in honour of Ilya Goldsheid's 70-th birthday
Queen Mary, University of London, December 18-22, 2017

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based on joint work with J. Shapiro

Outline

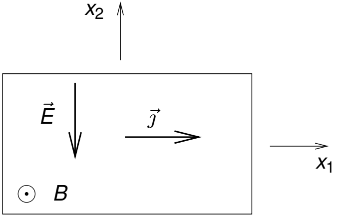
Some physics background first

- How it all began: Quantum Hall systems
- Topological insulators
- Bulk-edge correspondence
- The periodic table of topological matter

Chiral systems

- An experiment
- A chiral Hamiltonian and its indices
- Special cases

The Hall effect (1879)



B: magnetic field
$\vec{E}$: electric field
$\vec{\jmath}$: current density

Hall-Ohm law

$$\vec{\jmath} = \underline{\sigma}\vec{E}\,, \qquad \underline{\sigma} = \begin{pmatrix} \sigma_{\mathrm{D}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{D}} \end{pmatrix}$$

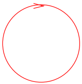
σ_{H}: Hall conductance
σ_{D}: dissipative conductance, ideally $= 0$

Bulk and edge transport

Cyclotron orbit

$\vec{E} = 0$

$\otimes \quad B$

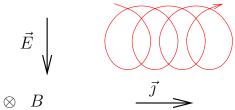


Bulk and edge transport

Cyclotron orbit

For $\vec{E} \neq 0$, two views on transport:

- Occurs throughout the sample (bulk): Cyclotron orbit drifting under a electric field $\vec{E}$



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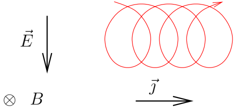
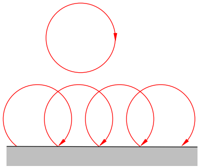


- Occurs along the edges: Skipping orbits



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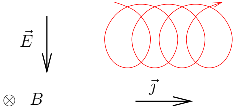

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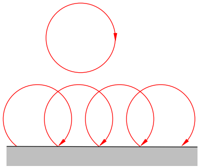
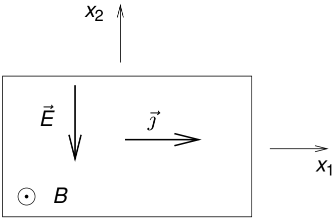


Two views: Complementary, but coexisting.

The experiment (von Klitzing, 1980)



Electron gas of
- density n
- Fermi energy μ

Hall-Ohm law

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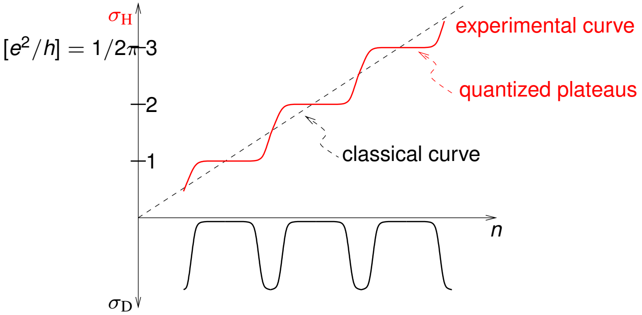

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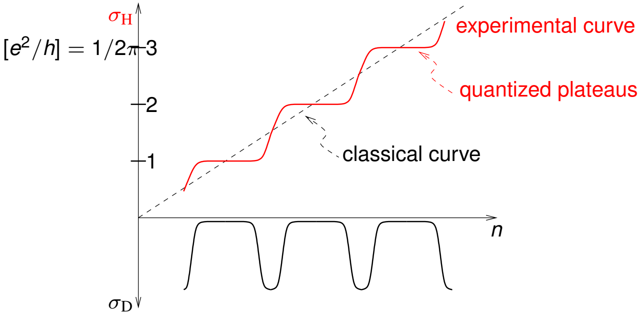
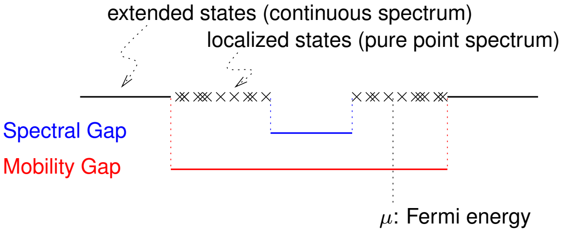


Width of plateaus increases with disorder

Spectral vs. Mobility Gap

The spectrum of a single-particle Hamiltonian



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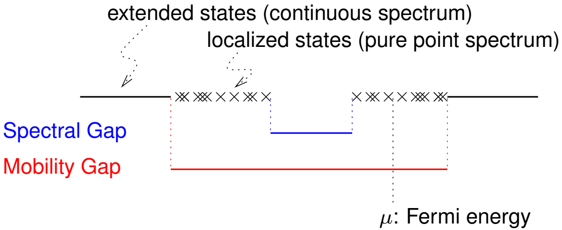

- (integrated) density of states $n(\mu)$ is constant for μ in a Spectral Gap, and strictly increasing otherwise

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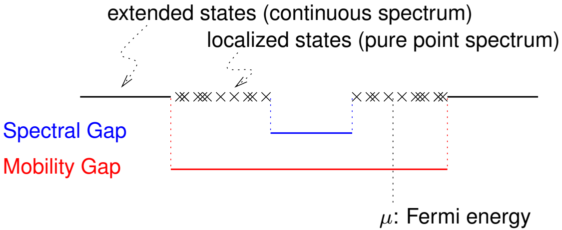

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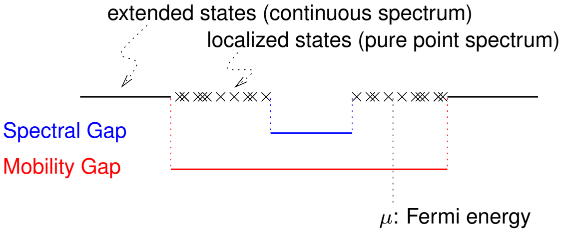

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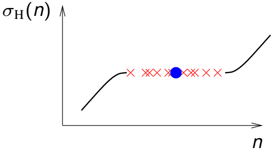
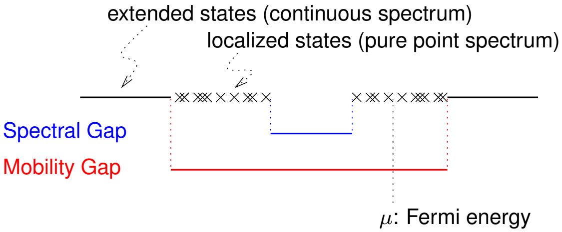


Plateaus arise because of a Mobility Gap only!

The role of disorder

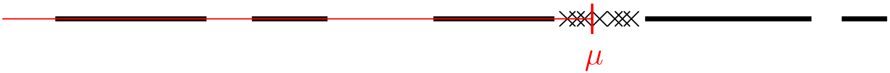
The spectrum of a single-particle Hamiltonian



- For a periodic (crystalline) medium:
  - Method of choice: Bloch theory and vector bundles (Thouless et al.)
  - Gap is spectral
- For a disordered medium:
  - Method of choice: Non-commutative geometry (Bellissard; Avron et al.)
  - Fermi energy may lie in a spectral gap or (better, and more generally) in a mobility gap.

Mobility gap, technically speaking

Hamiltonian H on $\ell^2(\mathbb{Z}^d)$
$P_\mu = I_{(-\infty,\mu)}(H)$: Fermi projection



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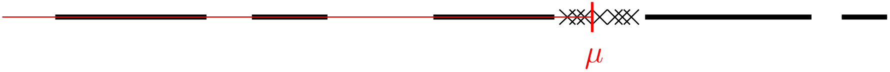

Assumption (Localization holds at Fermi energy). Fermi projection has strong off-diagonal decay:

$$P_\mu(x, x') \lesssim \mathrm{e}^{-\nu|x-x'|} \qquad (x, x' \in \mathbb{Z}^d)$$

(some $\nu > 0$)

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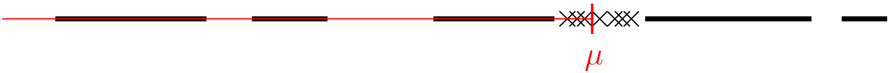

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$$\sup_{x' \in \mathbb{Z}^d} \mathrm{e}^{-\varepsilon |x'|} \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{\nu |x - x'|} |P_\mu(x, x')| < \infty$$

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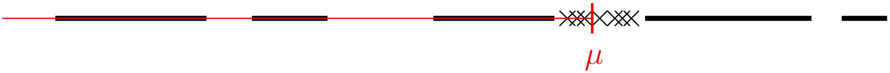

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- Proven in (virtually) all cases where localization is known.
- Trivially false for extended states at $E = \mu$.

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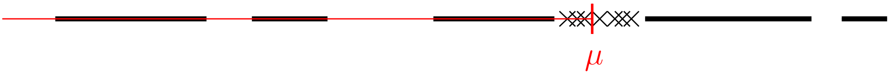

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Assumption $\mathcal{P}_\mu := I_{\{\mu\}}(H) = 0$

Topological insulators: definition stated

- Insulator in the Bulk: Excitation gap
  For independent electrons: spectral gap at Fermi energy μ

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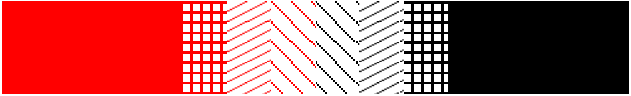
- Analogy: torus $\neq$ sphere (differ by genus)
- Refinement: The Hamiltonians enjoy a symmetry which is preserved under deformations.

Bulk-edge correspondence

Recall: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and respecting symmetries

Bulk-edge correspondence

Deformation as interpolation in physical space:



- Gap must close somewhere in between. Hence: Interface states at Fermi energy.

Bulk-edge correspondence

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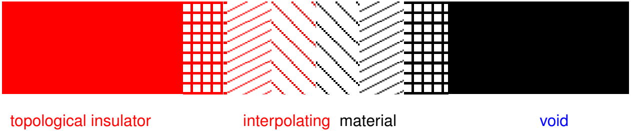

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Bulk-edge correspondence

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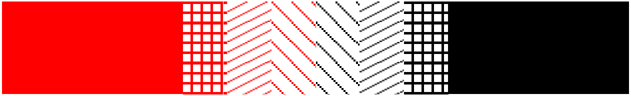

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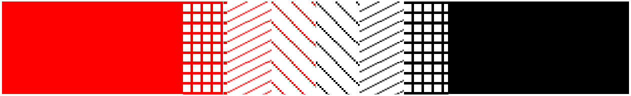
Bulk-edge correspondence

Deformation as interpolation in physical space:



topological insulator   interpolating material   void

- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
- Ordinary insulator $\rightsquigarrow$ void: Edge states
- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states. (But not conversely!)

Bulk-edge correspondence

In a nutshell: Termination of bulk of a topological insulator implies edge states

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- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

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- Express that property as an Index relating to the Bulk, resp. to the Edge.

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- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge.
- Bulk-edge duality: Can it be shown that the two indices agree? Can it be shown even in presence of just a mobility gap?

The periodic table of topological matter

Symmetry				*d*							
Class	Θ	Σ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Notation:

Θ time-reversal Σ charge conjugation Π combined

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AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
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C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

First version: Schnyder et al.; then Kitaev based on Altland-Zirnbauer; based on Bloch theory

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CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

By now: Non-commutative (bulk) index formulae have been found in many cases (Prodan, Schulz-Baldes)

Special case to be considered

Symmetry				*d*							
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CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Does the table survive disorder?

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A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Class A: Anderson localization (no topology)

Special case to be considered

Symmetry				d							
Class	Θ	Σ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
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DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Class A: Anderson localization (no topology)
Class AIII: Anderson localization, except possibly at one energy (topology rescued; by Ilya's methods)

An experiment: Amo et al.

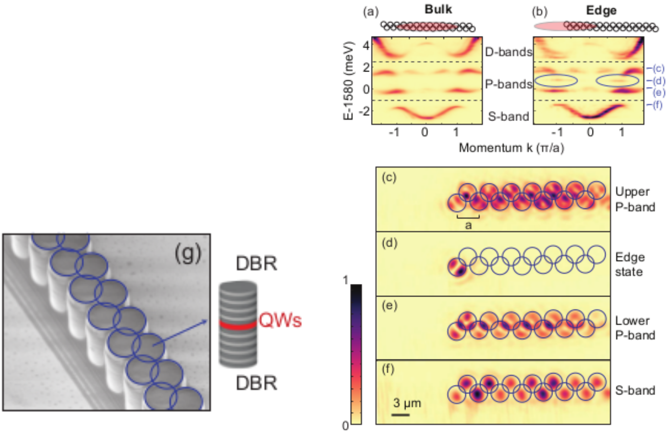


Figure: Zigzag chain of coupled micropillars and lasing modes

An experiment: Amo et al.

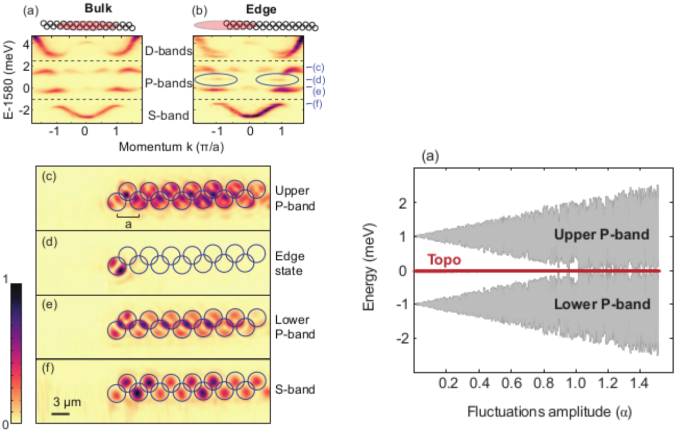
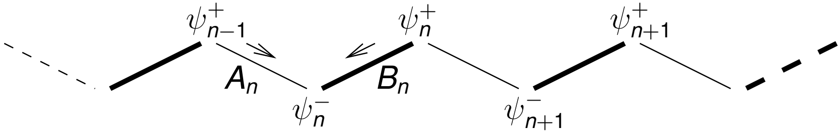


Figure: Lasing modes: bulk and edge

The Su-Schrieffer-Heeger model (1 dimensional)

Alternating chain with nearest neighbor hopping



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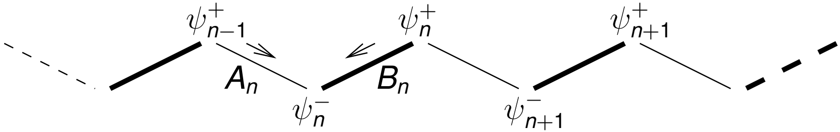

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n\in\mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n\psi^+_{n-1} + B_n\psi^+_n\,, \qquad (S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n$$

($A_n, B_n \in \mathrm{GL}(N)$ almost surely)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$H\psi = \lambda\psi \quad \Longrightarrow \quad H(\Pi\psi) = -\lambda(\Pi\psi)$$

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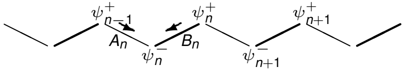


- Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n\,, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

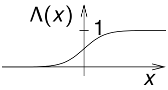
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2}\operatorname{tr}(\Pi\Sigma[\Lambda, \Sigma])$$



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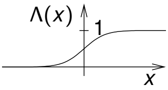

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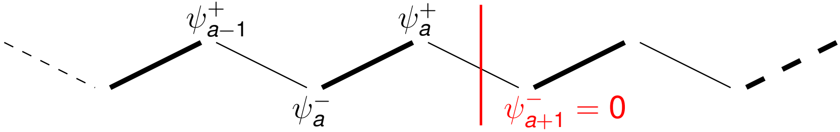
Equivalently

$$-\mathcal{N} = \operatorname{tr}(\Pi P_+[\Lambda, P_-]) + \operatorname{tr}(\Pi P_-[\Lambda, P_+])$$

using $P_+ := I_{(0,+\infty)}$, $P_- := I_{(-\infty,0)}$ and $\Sigma = P_+ - P_-$

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \le a$ (Dirichlet boundary condition $\psi_{a+1}^- = 0$). Chiral symmetry preserved.

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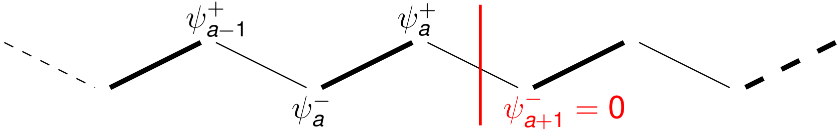

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Eigenspace $\operatorname{ran} \mathcal{P}_{0,a}$ invariant under Π.

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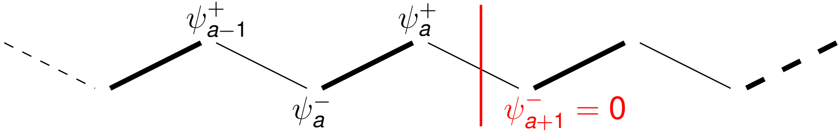

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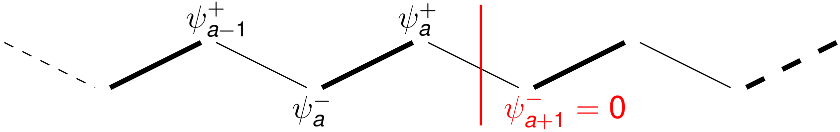

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Definition. The Edge index is

$$\mathcal{N}_a = \mathcal{N}^+_a - \mathcal{N}^-_a = \operatorname{tr}(\Pi \mathcal{P}_{0,a})$$

and can be shown to be independent of a. Call it $\mathcal{N}^\sharp$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

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Remark. Consider the dynamical system $A_n \psi^+_{n-1} + B_n \psi^+_n = 0$ with Lyaponov exponents

$$\gamma_1 \geq \ldots \geq \gamma_N$$

The assumption is satisfied if $\gamma_i \neq 0$; then $\mathcal{N}^\sharp = \sharp\{i \mid \gamma_i > 0\}$.

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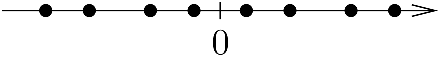
Lyapunov spectrum of the full chain has $2N$ exponents, spectrum is even ($N = 4$)

- at energy $\lambda \neq 0$ (simple spectrum)



  - Spectrum is simple because measure on transfer matrices is irreducible (cf. Goldsheid-Margulis)
  - so $\gamma = 0$ is not in the spectrum; localization follows

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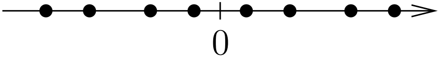

- At $\lambda = 0$ chains decouple: $\mathbb{C}^N \oplus 0$ and $0 \oplus \mathbb{C}^N$ are invariant subspaces

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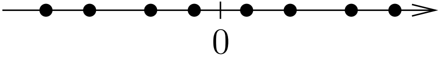
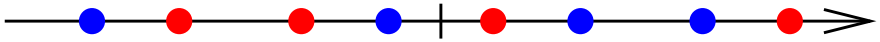
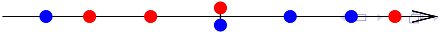


- of the upper ($+$) and lower ($-$) chains, at energy $\lambda = 0$



- at energy $\lambda = 0$ (phase boundary)



Proof of duality

Recall $\mathcal{N}_a = \mathrm{tr}(\Pi \mathcal{P}_{0,a})$

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\operatorname{tr}(\Pi\Lambda) = N \cdot \Big(\sum_{n \leq a} \Lambda(n)\Big) \operatorname{tr}_{\mathbb{C}^2} \Pi = 0$$

though $\|\Pi\Lambda\|_1 = \|\Lambda\|_1 \to \infty$, $(a \to +\infty)$

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So,

$$\mathrm{tr}(\Pi\Lambda) = \underbrace{\mathrm{tr}(\Pi\Lambda\mathcal{P}_{0,a})}_{\to\mathcal{N}^\sharp} + \underbrace{\mathrm{tr}(\Pi\Lambda P_{+,a}) + \mathrm{tr}(\Pi\Lambda P_{-,a})}_{\to\mathrm{tr}(\Pi P_-[\Lambda,P_+])+\mathrm{tr}(\Pi P_+[\Lambda,P_-])=-\mathcal{N}}$$

q.e.d.

Special case I: Spectral gap

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(General case is index theorem for non-Fredholm operator S_a)

Special case II: Translation invariant case

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$$\mathcal{N} = \frac{i}{2\pi} \oint dk \operatorname{tr} U(k)^* \partial_k U(k)$$

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- Equality: $(z = \mathrm{e}^{-\mathrm{i}k})$

$$\mathcal{N} = \frac{\mathrm{i}}{2\pi} \oint \frac{d \det U(k)}{\det U(k)} = \frac{1}{2\pi\mathrm{i}} \oint \frac{d \det S(z)}{\det S(z)} = \mathcal{N}_a$$

  by argument principle.

Summary

Elementary methods used to establish bulk-edge correspondence in simple models of topological insulators in presence of a mobility gap

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Thank you for your attention!

Best wishes, Ilya!