Exact solution of the classical dimer model on a triangular lattice

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Dimer Model

We consider the dimer model on a triangular lattice $\Gamma_{m,n} = (V_{m,n}, E_{m,n})$ on the torus $\mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}^2/(m\mathbb{Z} \times n\mathbb{Z})$ (periodic boundary conditions), where $V_{m,n}$ and $E_{m,n}$ are the sets of vertices and edges of $\Gamma_{m,n}$, respectively. It is convenient to consider $\Gamma_{m,n}$ as a *square lattice with diagonals*.
A *dimer* on $\Gamma_{m,n}$ is a set of two neighboring vertices $\langle x, y\rangle$ connected by an edge. A *dimer configuration* σ on $\Gamma_{m,n}$ is a set of dimers $\sigma = \{\langle x_i, y_i\rangle,\ i = 1, \dots, \frac{mn}{2}\}$ which cover $V_{m,n}$ without overlapping.

Dimer Configuration

An example of a dimer configuration is shown below. An obvious necessary condition for a configuration to exist is that at least one of m, n is even, and so we assume that m is even, $m = 2m_0$.

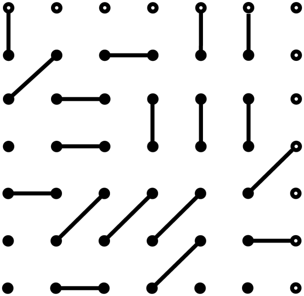


Figure: Example of a dimer configuration on a triangular 6×6 lattice on the torus.

Weights

To define a weight of a dimer configuration, we split the full set of dimers in a configuration σ into three classes: horizontal, vertical, and diagonal with respective weights $z_h, z_v, z_d > 0$. If we denote the total number of horizontal, vertical and diagonal dimers in σ by $N_h(\sigma)$, $N_v(\sigma)$, and $N_d(\sigma)$, respectively, then the *dimer configuration weight* is

$$w(\sigma) = \prod_{i=1}^{\frac{mn}{2}} w(x_i, y_i) = z_h^{N_h(\sigma)} z_v^{N_v(\sigma)} z_d^{N_d(\sigma)},$$

where $w(x_i, y_i)$ denotes the weight of the dimer $\langle x_i, y_i \rangle \in \sigma$.

Partition Function

We denote by $\Sigma_{m,n}$ the set of all dimer configurations on $\Gamma_{m,n}$. The *partition function* of the dimer model is given by

$$Z = \sum_{\sigma \in \Sigma_{m,n}} w(\sigma).$$

Notice that if all the weights are set equal to one, then Z simply counts the number of dimer configurations on $\Gamma_{m,n}$.

Kasteleyn's Formula

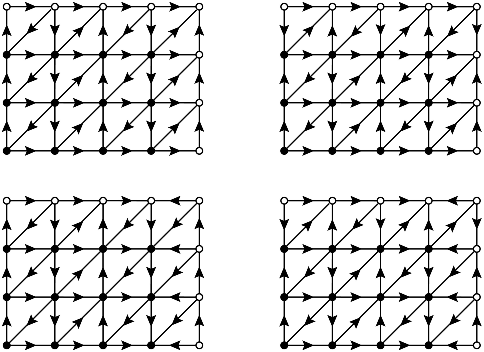
As shown by Kasteleyn, the partition function Z of the dimer model on the torus can be expressed in terms of the four Kasteleyn Pfaffians as

$$Z = \frac{1}{2}\left(-\mathrm{Pf}\, A_1 + \mathrm{Pf}\, A_2 + \mathrm{Pf}\, A_3 + \mathrm{Pf}\, A_4\right),$$

with periodic-periodic, periodic-antiperiodic, antiperiodic-periodic, and antiperiodic-antiperiodic boundary conditions in the x- and y-axis, respectively. The Kasteleyn's matrices A_i are *adjacency matrices with signs* determined by the *Kasteleyn's orientations*.

Kasteleyn's Orientations

We consider different orientations on the set of the edges $E_{m,n}$: O_1 (p-p), O_2 (p-a), O_3 (a-p), and O_4 (a-a).



All these orientations are *Kasteleyn orientations*, so that for any face the number of arrows on the boundary oriented *clockwise* is *odd*.

Kasteleyn's Sign Functions

With every orientation O_i we associate a sign function $\varepsilon_i(x, y)$, $x, y \in V_{m,n}$, defined as follows: if x and y are connected by an edge then

$$\varepsilon_i(x, y) = \begin{cases} 1, & \text{if the arrow in } O_i \text{ points from } x \text{ to } y, \\ -1, & \text{if the arrow in } O_i \text{ points from } y \text{ to } x, \end{cases}$$

and

$$\varepsilon_i(x, y) = 0, \quad \text{if } x \text{ and } y \text{ are not connected by an edge.}$$

Kasteleyn's Matrices A_i

To define the Kasteleyn matrices, consider any enumeration of the vertices, $V_{m,n} = \{x_1, \dots, x_{mn}\}$. Then the *Kasteleyn matrices A_i* are defined as

$$A_i = \big(a_i(x_j, x_k)\big)_{1 \le j,k \le mn}, \quad i = 1, 2, 3, 4,$$

with

$$a_i(x,y) = \begin{cases} \varepsilon_i(x,y) w(x,y), & \text{if } x \text{ and } y \text{ are connected by an edge,} \\ 0 & \text{otherwise,} \end{cases}$$

where $w(x,y) = z_h, z_v, z_d$ is the weight of the dimer $\langle x, y \rangle$ and ε_i is the sign function. The Kasteleyn matrices A_i are *antisymmetric*, so that $a_i(x_k, x_j) = -a_i(x_j, x_k)$.

Pfaffians

The *Pfaffian*, $\mathrm{Pf}\, A_i$, of the $mn \times mn$ antisymmetric matrix A_i, $i = 1, 2, 3, 4,$ is given by the formula,

$$\mathrm{Pf}\, A_i = \sum_{\pi} (-1)^{\pi} a_i(x_{p_1}, x_{p_2}) a_i(x_{p_3}, x_{p_4}) \cdots a_i(x_{p_{mn-1}}, x_{p_{mn}}),$$

where the sum is taken over all permutations,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & mn-1 & mn \\ p_1 & p_2 & p_3 & \cdots & p_{mn-1} & p_{mn} \end{pmatrix},$$

which satisfy the following restrictions:

(1) $p_{2\ell-1} < p_{2\ell}, \qquad 1 \le \ell \le \frac{mn}{2}$,

(2) $p_{2\ell-1} < p_{2\ell+1}, \quad 1 \le \ell \le \frac{mn}{2} - 1.$

Kasteleyn's Pfaffians

Such permutations are in a one-to-one correspondence with the dimer configurations, and

$$\mathrm{Pf}\, A_i = \sum_{\sigma \in \Sigma_{m,n}} (-1)^{\pi(\sigma)} w(\sigma) \prod_{\langle x,y \rangle \in \sigma} \varepsilon_i(x,y), \quad i = 1,2,3,4.$$

An important property of the *Kasteleyn Pfaffians* $\mathrm{Pf}\, A_i$ is that they do not depend on the enumeration of the vertices, $V_{m,n} = \{x_1, \dots, x_{mn}\}$.

Configuration Sign

The *sign of a configuration* σ, $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma; A_i)$, is the following expression:

$$\operatorname{sgn}(\sigma) = (-1)^{\pi(\sigma)} \prod_{\langle x,y\rangle \in \sigma} \varepsilon_i(x,y),$$

and the Pfaffian formula for a Kasteleyn matrix A_i can be rewritten as

$$\mathrm{Pf}\, A_i = \sum_{\sigma \in \Sigma_{m,n}} \operatorname{sgn}(\sigma) w(\sigma), \quad \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma; A_i).$$

Contours

Given two configurations σ and σ', we consider the double configuration $\sigma \cup \sigma'$, and we call it the *superposition* of σ and σ'. In $\sigma \cup \sigma'$, we define a *contour* to be a cycle consisting of alternating edges from σ and σ'. Each contour consists of an even number of edges. The superposition $\sigma \cup \sigma'$ is partitioned into disjoint contours $\{\gamma_k \ : \ k = 1, 2, \ldots, r\}$. We call a contour consisting of only two edges a *trivial contour*.

Standard Configuration

A *standard configuration* $\sigma_{\rm st}$ is defined as follows. Consider the lexicographic ordering of the vertices $(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Namely,

$$(i,j) = x_k, \quad k = jm + i + 1, \quad 1 \le k \le mn.$$

Then

$$\sigma_{\rm st} = \left\{ \langle x_{2l-1}, x_{2l} \rangle,\ l = 1, \ldots, \frac{mn}{2} \right\}.$$

Observe that the standard configurations consists of horizontal dimers only and

$$\operatorname{sgn}(\sigma_{\rm st}; A_i) = +1, \quad i = 1, 2, 3, 4,$$

because $\pi(\sigma_{\rm st}) = \mathrm{Id}$ and $\varepsilon_i(x_{2l-1}, x_{2l}) = +1$.

Configuration Sign Formula

Let σ, σ' be any two configurations and $\{\gamma_k \ : \ k = 1, 2, \ldots, r\}$ all contours of $\sigma \cup \sigma'$. Then

$$\mathrm{sgn}\,(\sigma; A_i) \cdot \mathrm{sgn}\,(\sigma'; A_i) = \prod_{k=1}^{r} \mathrm{sgn}\,(\gamma_k; O_i), \quad i = 1, 2, 3, 4,$$

with

$$\mathrm{sgn}\,(\gamma_k; O_i) = (-1)^{\nu_k(O_i)+1},$$

where $\nu_k(O_i)$ is the number of edges in γ_k oriented clockwise with respect to the orientation O_i.

Kasteleyn's Identities

As shown by Kasteleyn, the partition function Z can be decomposed as

$$Z = Z^{00} + Z^{10} + Z^{01} + Z^{11},$$

the four partition functions Z^{rs} corresponding to dimer configurations of the homology classes $(r,s) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the Pfaffians $\mathrm{Pf}\, A_i$ are expressed as

$$\mathrm{Pf}\, A_1 = Z^{00} - Z^{10} - Z^{01} - Z^{11}, \quad \mathrm{Pf}\, A_2 = Z^{00} - Z^{10} + Z^{01} + Z^{11},$$
$$\mathrm{Pf}\, A_3 = Z^{00} + Z^{10} - Z^{01} + Z^{11}, \quad \mathrm{Pf}\, A_4 = Z^{00} + Z^{10} + Z^{01} - Z^{11}$$

(*Kasteleyn's Identities*).

Pfaffian Sign Problem

The Kasteleyn's formula

$$Z = \frac{1}{2}\left(-\mathrm{Pf}\, A_1 + \mathrm{Pf}\, A_2 + \mathrm{Pf}\, A_3 + \mathrm{Pf}\, A_4\right),$$

is very powerful in the asymptotic analysis of the partition function as $m, n \to \infty$, because the absolute value of the Pfaffian of a square antisymmetric matrix A is determined by its determinant through the classical identity

$$(\mathrm{Pf}\, A)^2 = \det A.$$

The asymptotic behavior of $\det A_i$ as $m, n \to \infty$ can be analyzed by a diagonalization of the matrices A_i, and an obvious problem arises to determine the *sign* of the Pfaffians $\mathrm{Pf}\, A_i$.

Pfaffian Sign Theorem

Our main result with Elwood and Petrović is the following *Pfaffian Sign Theorem:*

Theorem (Bleher, Elwood, and Petrović)

Let $z_h, z_v, z_d > 0$*. Then*

$$\operatorname{Pf} A_1 < 0, \quad \operatorname{Pf} A_2 > 0, \quad \operatorname{Pf} A_3 > 0, \quad \operatorname{Pf} A_4 > 0.$$

Remarks

Kasteleyn considered the dimer model on the square lattice, which corresponds to the weight $z_d = 0$. He showed that in this case $\mathrm{Pf}\, A_1 = 0$, and he assumed that $\mathrm{Pf}\, A_i \geq 0$ for $i = 2, 3, 4$. Kenyon, Sun and Wilson established the sign of the Pfaffians $\mathrm{Pf}\, A_i$ for any *critical* dimer model on a lattice on the torus, including the square lattice. The dimer model on the triangular lattice is *not critical* and the result of Kenyon, Sun and Wilson is not applicable in this case. Different conjectures about the Pfaffian signs for the dimer model on a triangular lattice are stated, without proof, in the physical works of McCoy, Fendley–Moessner–Sondhi, and Izmailian–Kenna.

Ingredients of the Proof

The proof of Pfaffian Sign Theorem is based on the following three important ingredients:

1. The Kasteleyn identities.
2. The double product formula for $\det A_i$.
3. An asymptotic analysis of $\mathrm{Pf}\, A_i$ as one of the weights tends to zero.

Double Product Formula

We have that

$$\det A_i = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{n-1} S\left(\frac{j+\alpha_i}{m}, \frac{k+\beta_i}{n}\right),$$

where

$$S(x,y) = 4\left[z_h^2 \sin^2 2\pi x + z_v^2 \sin^2 2\pi y + z_d^2 \cos^2 (2\pi x + 2\pi y)\right]$$

and

$$\alpha_1 = \beta_1 = 0\,; \quad \alpha_2 = 0\,,\ \beta_2 = \frac{1}{2}\,; \quad \alpha_3 = \frac{1}{2}\,,\ \beta_3 = 0\,; \quad \alpha_4 = \beta_4 = \frac{1}{2}\,.$$

Spectral Function

The function

$$S(x,y) = 4\left[z_h^2 \sin^2 2\pi x + z_v^2 \sin^2 2\pi y + z_d^2 \cos^2(2\pi x + 2\pi y)\right]$$

is the *spectral function* of the dimer model. We have that if $z_h, z_v, z_d > 0$ then

$$S(x,y) > 0, \quad \forall x, y,$$

hence

$$\det A_i > 0.$$

As a consequence, we have that $\mathrm{Pf}\, A_i$ does not change the sign in the region $z_h, z_v, z_d > 0$; hence, it is sufficient to establish the sign of $\mathrm{Pf}\, A_i$ *at any point* of the region $z_h, z_v, z_d > 0$.

Positivity of $\mathrm{Pf}\, A_3$ and $\mathrm{Pf}\, A_4$

Let $z_h, z_v > 0$ and $z_d = 0$. Then $S(0,0) = 0$ and hence, by the Double Product Formula, $\det A_1 = 0$. This implies that

$$\mathrm{Pf}\, A_1 = Z^{00} - Z^{10} - Z^{01} - Z^{11} = 0,$$

hence

$$\mathrm{Pf}\, A_3 = Z^{00} + Z^{10} - Z^{01} + Z^{11} = 2Z^{10} + 2Z^{11} \geq 0,$$

but from the Double Product Formula we obtain that $\det A_3 > 0$, hence $\mathrm{Pf}\, A_3 > 0$ for all $z_h, z_v > 0$ and $z_d = 0$. By continuity, $\mathrm{Pf}\, A_3 > 0$ for all $z_h, z_v > 0$ and small $z_d > 0$, and hence for all $z_d > 0$ (since $\det A_3 > 0$). The same argument applies to $\mathrm{Pf}\, A_4$.

Positivity of $\mathrm{Pf}\, A_2$ and Negativity of $\mathrm{Pf}\, A_1$

The proof of positivity of $\mathrm{Pf}\, A_2$ and negativity of $\mathrm{Pf}\, A_1$ is more difficult and it depends on the values of m and n modulo 4. To prove the negativity of $\mathrm{Pf}\, A_1$, we consider the following cases:

1. $m \equiv 2 \pmod 4$ or $n \equiv 2 \pmod 4$.
2. $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$.
3. $m \equiv 4 \pmod 4$ and $n \equiv 4 \pmod 4$.

The first case can be analyzed with the help of the Kasteleyn identities. The second and third cases are more difficult.

Case 2: $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$

In Case 2 we prove the following result. Let $z_h = 1$ and $z_v = 0$. Then as $z_d \to +0$,

$$\mathrm{Pf}\, A_1 = -2 \left(\frac{m}{2}\right)^n z_d^n (1 + \mathcal{O}(z_d)).$$

This implies that $\mathrm{Pf}\, A_1 < 0$ for $z_h = 1$, $z_v = 0$ and sufficiently small z_d, and hence $\mathrm{Pf}\, A_1 < 0$ for all $z_h, z_v, z_d > 0$. The asymptotics of $\mathrm{Pf}\, A_1$ shows that all terms with less than n diagonal dimers *cancel out* in the Pfaffian.

Case 3: $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$

In Case 3 we prove the following result. Let $z_h = 1$ and $0 < z_v \le z_d^2$. Then as $z_d \to +0$,

$$\mathrm{Pf}\, A_1 = -n^2 \left(\frac{m}{2}\right)^n z_v^2 z_d^{n-2} \left(1 + \mathcal{O}\left(z_d\right)\right).$$

This implies that $\mathrm{Pf}\, A_1 < 0$ for $z_h = 1$, $z_v = z_d^2$ and sufficiently small z_d, and hence $\mathrm{Pf}\, A_1 < 0$ for all $z_h, z_v, z_d > 0$.

Asymptotics of The Partition Function

Theorem

Suppose that $m, n \to \infty$ in such a way that $C_1 \le \frac{m}{n} \le C_2$ for some positive constants $C_2 > C_1$. Then for some $c > 0$,

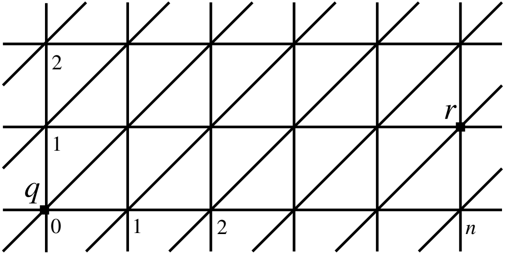
$$Z = 2e^{\frac{1}{2}mnF}\left(1 + \mathcal{O}\left(e^{-c(m+n)}\right)\right),$$

where $F = \ln 2 + \int\limits_0^1 \int\limits_0^1 f(x,y)\,dx\,dy$, *and*

$$f(x,y) = \frac{1}{2}\ln\left[z_h^2 \sin^2(2\pi x) + z_v^2 \sin^2(2\pi y) + z_d^2 \cos^2(2\pi x + 2\pi y)\right].$$

Project 2: Exact Solution for the Monomer–Monomer Correlation Function

This is a joint project with Estelle Basor (American Institute of Mathematics). We consider the classical dimer model on a triangular lattice. Again, it is convenient to view the triangular lattice as a square lattice with diagonals:



Main Goal

Our main goal is to calculate an asymptotic behavior as $n \to \infty$ of the *monomer-monomer correlation function* $K_2(n)$ between two vertices q and r that are n spaces apart in adjacent rows, in the thermodynamic limit (infinite volume).
We consider the dimer weights

$$w_h = w_v = 1, \qquad w_d = t > 0.$$

When $t = 1$, the dimer model is symmetric, and when $t = 0$, it reduces to the dimer model on the square lattice, hence changing t from 0 to 1 gives a deformation of the dimer model on the square lattice to the symmetric dimer model on the triangular lattice.

Monomer-monomer correlation function as a block Toeplitz determinant

Our starting point is a determinantal formula for $K_2(n)$ (*Fendley–Moessner–Sondhi–Basor–Ehrhardt*):

$$K_2(n) = \frac{1}{2}\sqrt{\det T_n(\phi)},$$

where $T_n(\phi)$ is the finite block Toeplitz matrix,

$$T_n(\phi) = (\phi_{j-k}), \quad 0 \le j, k \le n-1,$$

where

$$\phi_k = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{ix})e^{-ikx}\,dx.$$

Block Symbol $\phi(e^{ix})$

The 2×2 matrix symbol $\phi(e^{ix})$ is

$$\phi(e^{ix}) = \sigma(e^{ix}) \begin{pmatrix} p(e^{ix}) & q(e^{ix}) \\ q(e^{-ix}) & p(e^{-ix}) \end{pmatrix},$$

with

$$\sigma(e^{ix}) = \frac{1}{(1 - 2t\cos x + t^2)\sqrt{t^2 + \sin^2 x + \sin^4 x}}$$

and

$$p(e^{ix}) = (t\cos x + \sin^2 x)(t - e^{ix}),$$
$$q(e^{ix}) = \sin x(1 - 2t\cos x + t^2).$$

BOCG Type Formula

To evaluate the asymptotics of $\det T_n(\phi)$ as $n \to \infty$ we use a *Borodin–Okounkov–Case–Geronimo* (BOCG) type formula for block Toeplitz determinants. For any matrix-valued 2π-periodic matrix-valued function $\varphi(e^{ix})$ consider the corresponding semi-infinite matrices, Toeplitz and Hankel,

$$T(\varphi) = (\varphi_{j-k})_{j,k=0}^{\infty}\,; \qquad H(\varphi) = (\varphi_{j+k+1})_{j,k=0}^{\infty}\,,$$

where

$$\varphi_k = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{ix})e^{-ikx}dx$$

BOCG Type Formula

Let $\psi(e^{ix}) = \phi^{-1}(e^{ix})$, where the matrix symbol $\phi(e^{ix})$ was introduced before, and the inverse is the matrix inverse. Then the following BOCG type formula holds:

$$\det T_n(\phi) = \frac{E(\psi)}{G(\psi)^n} \det\left(I - \Phi\right),$$

where $\det\left(I - \Phi\right)$ is the Fredholm determinant with

$$\Phi = H\left(e^{-inx}\psi(e^{ix})\right) T^{-1}\left(\psi(e^{-ix})\right) H\left(e^{-inx}\psi(e^{-ix})\right) T^{-1}\left(\psi(e^{ix})\right).$$

In our case $G(\psi) = 1$ and

$$E(\psi) = \frac{t}{2t(2+t^2) + (1+2t^2)\sqrt{2+t^2}}$$

(the *Basor–Ehrhardt* formula).

Order Parameter

The Basor–Ehrhardt formula implies that the order parameter is equal to

$$K_2(\infty) := \lim_{n\to\infty} K_2(n) = \frac{1}{2}\sqrt{E(\psi)}$$
$$= \frac{1}{2}\sqrt{\frac{t}{2t(2+t^2)+(1+2t^2)\sqrt{2+t^2}}}.$$

Our goal is to evaluate an asymptotic behavior of $K_2(n)$ as $n \to \infty$. The problem reduces to evaluating an asymptotic behavior of the Fredholm determinant $\det(I - \Phi)$, because

$$K_2(n) = K_2(\infty)\sqrt{\det(I - \Phi)}.$$

The Wiener–Hopf Factorization of $\phi(z)$

To evaluate $\det(I-\Phi)$ we need to invert the semi-infinite Toeplitz matrices $T^{-1}(\psi(e^{ix}))$ and to do so we use the *Wiener–Hopf factorization* of the symbol ϕ. Let $z = e^{ix}$. Denote

$$\pi(z) = \begin{pmatrix} p(z) & q(z) \\ q(z^{-1}) & p(z^{-1}) \end{pmatrix},$$

so that

$$\phi(z) = \sigma(z)\pi(z),$$

where

$$\sigma(z) = \frac{1}{(1-2t\cos x + t^2)\sqrt{t^2+\sin^2 x + \sin^4 x}}$$

is a scalar function.

The Wiener–Hopf factorization

Our goal is to factor the matrix-valued symbol $\phi(z)$ as $\phi(z) = \phi_+(z)\phi_-(z)$, where $\phi_+(z)$ and $\phi_-(z^{-1})$ are *analytic invertible* matrix valued functions on the disk $D = \{z \mid |z| \le 1\}$. Denote

$$\tau = \frac{1}{t} \,.$$

We start with an explicit factorization of the function $t^2 + \sin^2 x + \sin^4 x$.

Factorization of $t^2 + \sin^2 x + \sin^4 x$ and the Numbers $\eta_{1,2}$

We have that

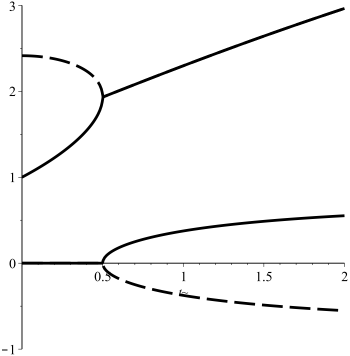
$$t^2+\sin^2 x+\sin^4 x = \frac{1}{16\eta_1^2\eta_2^2}\left(z^{-2}-\eta_1^2\right)\left(z^{-2}-\eta_2^2\right)\left(z^{2}-\eta_1^2\right)\left(z^{2}-\eta_2^2\right),$$

where

$$\eta_{1,2} = \frac{1}{\sqrt{2\pm\mu-2\sqrt{1-t^2\pm\mu}}}, \quad \mu=\sqrt{1-4t^2}.$$

The numbers $\eta_{1,2}$ are positive for $0 \le t \le \frac{1}{2}$ and complex conjugate for $t > \frac{1}{2}$.

Graphs of η_1, η_2



The graphs of $|\eta_1(t)|$ (dashed line), $|\eta_2(t)|$ (solid line), the upper graphs, and $\arg\eta_1(t)$ (dashed line), $\arg\eta_2(t)$ (solid line), the lower graphs

Wiener–Hopf Factorization

Theorem 1. We have the Wiener–Hopf factorization:

$$\phi(z) = \phi_+(z)\phi_-(z),$$

where

$$\phi_+(z) = A(z)\Psi(z), \quad \phi_-(z) = \Psi^{-1}(z^{-1}),$$

with

$$A(z) = \frac{\tau}{z-\tau},$$

and

$$\Psi(z) = \frac{1}{\sqrt{f(z)}} D_0(z)P_1D_1(z)P_2D_2(z)P_3D_3(z)P_4D_4(z)P_5,$$

with

Wiener–Hopf Factorization

$$f(z) = \frac{(z^2 - \eta_1^2)(z^2 - \eta_2^2)}{4\eta_1\eta_2}$$

and

$$D_0(z) = \begin{pmatrix} 1 & 0 \\ 0 & z - \tau \end{pmatrix},$$

$$D_1(z) = \begin{pmatrix} z - \eta_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2(z) = \begin{pmatrix} z + \eta_1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_3(z) = \begin{pmatrix} z - \eta_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_4(z) = \begin{pmatrix} 1 & 0 \\ 0 & z + \eta_2 \end{pmatrix},$$

and

$$P_j = \begin{pmatrix} 1 & p_j \\ 0 & 1 \end{pmatrix}, \quad j = 1, 2, 3, 5; \quad P_4 = \begin{pmatrix} 1 & 0 \\ p_4 & 1 \end{pmatrix}.$$

Wiener–Hopf Factorization

Here

$$p_1 = \frac{i\left[\tau(\eta_1^2-1)^2 - 2\eta_1(\eta_1^2+1)\right]}{2(\eta_1^2-1)}\,, \quad p_2 = -\frac{i(\eta_1^2+1)}{\eta_1^2-1}\,,$$

$$p_3 = \frac{i\tau(\eta_1+1)}{2\eta_1}\,, \quad p_4 = -\frac{2i\eta_1\eta_2}{\tau}\,, \quad p_5 = -\frac{i\tau}{2\eta_1}\,.$$

Idea of the proof

The idea of the proof goes back to the works of *McCoy and Wu* on the Ising model, and even before to the works of *Hopf* and *Grothendieck*.

Let us recall that $\phi(z) = \sigma(z)\pi(z)$, where $\sigma(z)$ is a scalar function. The difficult part is to factor $\pi(z)$. To factor $\pi(z)$ we use a *decreasing power algorithm*. In this algorithm at every step we make a substitution decreasing the power in z of the matrix entries under consideration.

Minus-Plus Factorization of $\phi(z)$

Applying the symmetry relation,

$$\phi(z) = \sigma_3 \phi^{\mathrm{T}}(z) \sigma_3,$$

to the plus-minus factorization of $\phi(z)$,

$$\phi(z) = \phi_+(z)\phi_-(z),$$

we obtain a minus-plus factorization of $\phi(z)$:

$$\phi(z) = \theta_-(z)\theta_+(z),$$

where

$$\theta_-(z) = \sigma_3 \phi_-^{\mathrm{T}}(z), \quad \theta_+(z) = \phi_+^{\mathrm{T}}(z)\sigma_3.$$

A Useful Formula for the Fredholm Determinant $\det(I-\Phi)$

Our goal is to evaluate the Fredholm determinant $\det(I-\Phi)$, with

$$\Phi = H\big(e^{-inx}\psi(e^{ix})\big)\,T^{-1}\big(\psi(e^{-ix})\big)\,H\big(e^{-inx}\psi(e^{-ix})\big)\,T^{-1}\big(\psi(e^{ix})\big).$$

This Φ is not very handy for an asymptotic analysis. We have another useful representation of $\det(I-\Phi)$:

$$\det(I-\Phi) = \det(I-\Lambda),$$

where

$$\Lambda = H(z^{-n}\alpha)H(z^{-n}\beta)$$

with

$$\alpha(z) = \phi_-(z)\theta_+^{-1}(z), \quad \beta(z) = \theta_-^{-1}(z^{-1})\phi_+(z^{-1}).$$

The Matrix Elements of the Matrix Λ

The matrix elements of the matrix Λ are

$$\Lambda_{jk} = \sum_{a=0}^{\infty} \alpha_{j+n+a+1} \beta_{k+n+a+1},$$

where

$$\alpha_k = \frac{1}{2\pi} \int_0^{2\pi} \alpha(e^{ix}) e^{-ikx} dx, \quad \beta_k = \frac{1}{2\pi} \int_0^{2\pi} \beta(e^{ix}) e^{-ikx} dx.$$

We point out that this representation allows for a more direct computation of the determinant of interest without the more complicated formula involving the operator inverses.

Asymptotics of the monomer-monomer correlation function for $0 < t < \frac{1}{2}$.

Theorem 2. Let $0 < t < \frac{1}{2}$. Then as $n \to \infty$,

$$K_2(n) = K_2(\infty)\left[1 - \frac{e^{-2n\ln\eta_2}}{2n}\left(C_1 + (-1)^{n+1}C_2 + \mathcal{O}(n^{-1})\right)\right],$$

with some explicit $C_1, C_2 > 0$.

Corollary. This gives that the *correlation length* is equal to

$$\xi = \frac{1}{2\ln\eta_2}\,.$$

As $t \to 0$,

$$\xi = \frac{1}{2t} + \mathcal{O}(1)\,.$$

Asymptotics of the monomer-monomer correlation function for $\frac{1}{2} < t < 1$.

If $t > \frac{1}{2}$, then η_1, η_2 are complex conjugate numbers,

$$\eta_1 = e^{s-i\theta}, \quad \eta_2 = e^{s+i\theta};$$
$$s = \ln|\eta_1| = \ln|\eta_2| > 0; \quad 0 < \theta < \frac{\pi}{4}.$$

Main Results

Theorem 3. Assume that $\frac{1}{2} < t < 1$. Then as $n \to \infty$,

$$K_2(n) = K_2(\infty)\Big[1 - \frac{e^{-2ns}}{2n}\Big(C_1\cos(2\theta n + \varphi_1) + C_2(-1)^n\cos(2\theta n + \varphi_2) + C_3 + C_4(-1)^n\Big) + \mathcal{O}(n^{-1})\Big],$$

with $s = \ln|\eta_1| = \ln|\eta_2|$, $\theta = |\arg\eta_1| = |\arg\eta_2|$, and explicit C_1, C_2, C_3 , C_4, φ_1, φ_2.

References

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The End

Thank you!

Happy Birthday, Ilya!