Explicit representations of maximal subgroups of the Monster

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Abstract

Most of the maximal subgroups of the Monster are now known, but in many cases they are hard to calculate in. We produce explicit 'small' representations of all the maximal subgroups which are not 2-local. The representations we construct are available on the World Wide Web at http://brauer.maths.qmul.ac.uk/Atlas/.

1 The maximal subgroups of the Monster

By the work of several authors [13, 14, 17, 12, 5, 9, 7, 8] over many years, the classification of the maximal subgroups of the Monster into conjugacy classes is almost complete. There are 43 conjugacy classes of known maximal subgroups, as shown in Table 1, and any other maximal subgroup is almost simple, with socle isomorphic to one of the following simple groups: $\mathrm{L}_2(13)$, $\mathrm{U}_3(4)$, $\mathrm{U}_3(8)$, $\mathrm{Sz}(8)$.

Calculating in the Monster itself is hard (see [11]), but for some purposes we would like to be able to calculate in some maximal subgroup instead. In many cases, this will be a lot easier. In practice, some of the 2-local subgroups seem to be too big for easy computations, but most of the other subgroups can be brought within range.

Our aim is to provide representations of these maximal subgroups. The bulk of the work described in this paper is devoted to constructing the large p-local subgroups (for p odd), in particular four maximal 3-local subgroups and two tricky cases for $p = 5$. For completeness, we also include constructions of all the other known maximal subgroups which are not 2-local. In some cases these groups are relatively easy to construct, and we do not claim anything new. Note that Richardson [15] has provided alternative constructions of three groups of type $\mathrm{N}_{\mathbb{M}}(p^2)$. Also, in some cases the precise structures are either not available, or not proved, in the literature. In these cases, we have remedied this deficiency, but of course many of these results are also not new. The representations we construct are available on the World Wide Web at http://brauer.maths.qmul.ac.uk/Atlas/.

Table 1: Known maximal subgroups of the Monster

Description	Shapes of maximal subgroups
2-locals	$2^{\cdot}\mathbb{B}$, $2^{2\cdot 2}\mathrm{E}_6(2){:}\mathrm{S}_3$, $2_+^{1+24\cdot}\mathrm{Co}_1$
	$2^2.2^{11}.2^{22\cdot}(\mathrm{S}_3 \times \mathrm{M}_{24})$, $2^3.2^6.2^{12}.2^{18\cdot}(\mathrm{L}_3(2) \times 3\mathrm{S}_6)$
	$2^5.2^{10}.2^{20\cdot}(\mathrm{S}_3 \times \mathrm{L}_5(2))$, $2^{10}.2^{16\cdot}\mathrm{O}_{10}^+(2)$
3-locals	$3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$, $3_+^{1+12\cdot}2^{\cdot}\mathrm{Suz}{:}2$, $3^2.3^5.3^{10\cdot}(\mathrm{M}_{11} \times 2\mathrm{S}_4^+)$
	$3^3.3^2.3^6.3^6{:}(\mathrm{L}_3(3) \times \mathrm{SD}_{16})$, $3^{\cdot}\mathrm{Fi}_{24}$
local extraspecial type	$5_+^{1+6}{:}2\mathrm{J}_2.4$, $7_+^{1+4}{:}(3 \times 2\mathrm{S}_7^-)$, $13_+^{1+2}{:}(3 \times 4\mathrm{S}_4)$
local affine type	$5^4{:}(3 \times 2^{\cdot}\mathrm{L}_2(25)){:}2_2$, $7^2{:}2^{\cdot}\mathrm{L}_2(7)$, $11^2{:}(5 \times 2\mathrm{A}_5)$
	$13^2{:}2^{\cdot}\mathrm{L}_2(13){:}4$, $41{:}40$
local subdirect products	$(3^2{:}2 \times \mathrm{O}_8^+(3))^{\cdot}\mathrm{S}_4$, $\mathrm{S}_3 \times \mathrm{Th}$, $(\mathrm{D}_{10} \times \mathrm{HN})^{\cdot}2$
	$(5^2{:}(4 \circ \mathrm{Q}_8) \times \mathrm{U}_3(5)){:}\mathrm{S}_3$, $(7{:}3 \times \mathrm{He}){:}2$
	$(7^2{:}(3 \times 2\mathrm{A}_4) \times \mathrm{L}_2(7)).2$, $(13{:}6 \times \mathrm{L}_3(3))^{\cdot}2$,
other local subgroups	$7^2.7.7^2{:}\mathrm{GL}_2(7)$, $5^{3\cdot}5^{3\cdot}(2 \times \mathrm{L}_3(5))$, $5^2.5^2.5^4{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$
non-local subdirect products	$(\mathrm{L}_2(11) \times \mathrm{M}_{12}){:}2$, $(\mathrm{A}_5 \times \mathrm{A}_{12}){:}2$, $\mathrm{M}_{11} \times \mathrm{A}_6{}^{\cdot}2^2$,
	$(\mathrm{L}_3(2) \times \mathrm{S}_4(4){:}2)^{\cdot}2$, $(\mathrm{A}_5 \times \mathrm{U}_3(8){:}3_1){:}2$
subwreath products	$\mathrm{A}_6{}^{3\cdot}(2 \times \mathrm{S}_4)$, $(\mathrm{A}_7 \times (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2){:}2$, $\mathrm{S}_5{}^3{:}\mathrm{S}_3$, $\mathrm{L}_2(11)^2{:}4$
almost simple subgroups	$\mathrm{L}_2(71)$, $\mathrm{L}_2(59)$, $\mathrm{L}_2(29){:}2$, $\mathrm{L}_2(19){:}2$

We roughly divide the subgroups into the following categories: (i) almost simple groups, (ii) subdirect products and subwreath products, (iii) easy local subgroups, and (iv) hard local subgroups. The first category we leave for the Web-Atlas [18]. The second and third categories are easy to construct from scratch, provided we can determine the precise isomorphism type of the group to be constructed. In the fourth category we put most of the 3-local subgroups, and some of the 5- and 7-locals. In these cases we adopted a variety of techniques, as explained below.

Notation. Notation for groups largely follows the ATLAS [4]. In particular $\mathrm{O}_n^\varepsilon(q)$ denotes the *simple* orthogonal group, often denoted $\mathrm{P}\Omega_n^\varepsilon(q)$ elsewhere in the literature. We adopt Griess's suggestion of writing $G\frac{1}{2}$ to denote a subgroup of index 2 in G, and particularly $(G \times H)\frac{1}{2}$ for a subgroup of index 2 in $G \times H$ which is not of the form $G\frac{1}{2} \times H$ or $G \times H\frac{1}{2}$. We further generalise by writing $(G \times H)\frac{1}{K}$ for a diagonal product

$$G \triangle_K H := \{\, (g, h) \in G \times H \mid \phi(g) = \psi(h) \in K \,\}$$

for fixed epimorphisms $\phi : G \to K$ and $\psi : H \to K$. Note that if K has trivial outer automorphism group then the isomorphism type of $(G \times H)\frac{1}{K}$ is determined by $\ker \phi$ and $\ker \psi$; this is the situation that pertains in this paper.

Simple modules (and their associated irreducible characters) are denoted by their degrees, often followed by distinguishing letters and/or signs. In particular, 1^+ denotes the trivial module or character, and 1^- denotes the linear character (and corresponding module) with kernel G for a group of shape $G.2$ or $G \times 2$. For all such groups we write $M^- = M^+ \otimes 1^-$, and in the case of

$G \times 2$ the central involution is in the kernel of M^+. If U and V are modules, $U \cdot V$ denotes a uniserial module with constituents U and V, such that U is a submodule.

Sometimes we combine the group and module notations, so that for example $5^{3a^-}{:}(\mathrm{L}_3(5) \times 2)$ denotes the split extension of the $3a^-$ module over GF(5) for $\mathrm{L}_3(5) \times 2$ by the group $\mathrm{L}_3(5) \times 2$ itself.

2 Almost simple subgroups

In this section we include the maximal subgroups $\mathrm{L}_2(71)$, $\mathrm{L}_2(59)$, $\mathrm{L}_2(29){:}2$ and $\mathrm{L}_2(19){:}2$, which can easily be constructed as permutation groups on 72, 60, 30 and 20 points respectively. We may as well include also $3{}^{\cdot}\mathrm{Fi}_{24} \cong 3{}^{\cdot}\mathrm{Fi}_{24}'{:}2$, which is available in [18] both as permutations on 920808 points and in 1566 dimensions over GF(2). For the sake of completeness, note that [18] also contains representations of all extensions of $\mathrm{L}_2(13)$, $\mathrm{U}_3(4)$, $\mathrm{U}_3(8)$ or $\mathrm{Sz}(8)$ by outer automorphisms.

3 Subdirect products

The groups treated in this section have shapes as follows: $(3^2{:}2 \times \mathrm{O}_8^+(3)){}^{\cdot}\mathrm{S}_4$, $\mathrm{S}_3 \times \mathrm{Th}$, $(\mathrm{D}_{10} \times \mathrm{HN}){}^{\cdot}2$, $(5^2{:}(4 \circ \mathrm{Q}_8) \times \mathrm{U}_3(5)){:}\mathrm{S}_3$, $(7{:}3 \times \mathrm{He}){:}2$, $(7^2{:}(3 \times 2\mathrm{A}_4) \times \mathrm{L}_2(7)).2$, $(\mathrm{L}_2(11) \times \mathrm{M}_{12}){:}2$, $(13{:}6 \times \mathrm{L}_3(3)){}^{\cdot}2$, $(\mathrm{A}_5 \times \mathrm{A}_{12}){:}2$, $\mathrm{M}_{11} \times \mathrm{A}_6{}^{\cdot}2^2$, $(\mathrm{L}_3(2) \times \mathrm{S}_4(4){:}2){}^{\cdot}2$. and $(\mathrm{A}_5 \times \mathrm{U}_3(8){:}3_1){:}2$. In these cases the only real difficulty is in deciding exactly which subgroup of the corresponding direct product we need to take. The direct product itself is of course trivial to construct from representations of the two factors.

Two of these groups are actually direct products, namely $\mathrm{S}_3 \times \mathrm{Th}$ and $\mathrm{M}_{11} \times \mathrm{A}_6{}^{\cdot}2^2$. The former is best represented in 250 dimensions over GF(2), as the direct sum of the 2-dimensional representation of $\mathrm{S}_3 \cong \mathrm{GL}_2(2)$ and the 248-dimensional representation of the Thompson simple group. The other case can be represented as a permutation group on 21 points, as the disjoint union of the representations of M_{11} on 11 points and $\mathrm{A}_6{}^{\cdot}2^2 \cong \mathrm{P\Gamma L}_2(9)$ on the 10-point projective line.

Three more of the groups are normalisers of cyclic groups, and their structure is well-known. These are $(\mathrm{D}_{10} \times \mathrm{HN}){}^{\cdot}2 \cong (5{:}4 \times \mathrm{HN}{:}2)\frac{1}{2}$, $(7{:}3 \times \mathrm{He}){:}2 \cong (7{:}6 \times \mathrm{He}{:}2)\frac{1}{2}$, and $(13{:}6 \times \mathrm{L}_3(3)){}^{\cdot}2 \cong (13{:}12 \times \mathrm{L}_3(3){:}2)\frac{1}{2}$. To illustrate the procedure here we take the last case, and represent the Frobenius group 13:12 as permutations on 13 points in the natural way, and represent $\mathrm{L}_3(3){:}2$ as permutations on the 26 points and lines of the projective plane of order 3. If we take standard generators c and d for $\mathrm{L}_3(3){:}2$, then they are both in the outer half. So let us take generators a and b in the outer half of 13:12 (so that they each have order 4 or 12). It follows that ac and bd generate the required subgroup of index 2 in $13{:}12 \times \mathrm{L}_3(3){:}2$. A similar construction gives $(7{:}3 \times \mathrm{He}){:}2$ as permutations on $7 + 2058$ points, and $(\mathrm{D}_{10} \times \mathrm{HN}){}^{\cdot}2$ in a matrix representation of dimension 135 over GF(5). In this last case, we represent the Frobenius group 5:4 as a group of upper-triangular 2×2 matrices.

The structure of the five groups

$$
\begin{aligned}
(3^2{:}2 \times \mathrm{O}_8^+(3))^{\cdot}\mathrm{S}_4 &\cong (3^2{:}\mathrm{GL}_2(3) \times \mathrm{Aut}(\mathrm{O}_8^+(3)))\frac{1}{\mathrm{S}_4}\\
(\mathrm{L}_2(11) \times \mathrm{M}_{12}){:}2 &\cong (\mathrm{L}_2(11){:}2 \times \mathrm{M}_{12}{:}2)\frac{1}{2}\\
(\mathrm{A}_5 \times \mathrm{A}_{12}){:}2 &\cong (\mathrm{S}_5 \times \mathrm{S}_{12})\frac{1}{2}\\
(\mathrm{L}_3(2) \times \mathrm{S}_4(4){:}2)^{\cdot}2 &\cong (\mathrm{L}_3(2){:}2 \times \mathrm{S}_4(4){:}4)\frac{1}{2}\\
(\mathrm{A}_5 \times \mathrm{U}_3(8){:}3_1){:}2 &\cong (\mathrm{S}_5 \times \mathrm{U}_3(8){:}6)\frac{1}{2}
\end{aligned}
$$

is given by Norton in [13]. Thus these groups may be constructed easily from representations of the factors of the subdirect product. A few of these were not already available in the Web-Atlas [18], so we had to make them. These were $\mathrm{Aut}(\mathrm{O}_8^+(3))$, $\mathrm{U}_3(8){:}6$, and $\mathrm{S}_4(4){:}4$ which we made as permutations on 3360, 513 and 170 points respectively.

In fact, $\mathrm{U}_3(8){:}6 \cong \mathrm{P}\Sigma\mathrm{U}_3(8)$ is available in Magma as a permutation group on 513 points (the cosets of the Borel subgroup). The other two groups include exceptional graph automorphisms, and are not quite so easy to make. We start with the natural module of dimension 8 for $2^{\cdot}\mathrm{O}_8^+(3)$ over GF(3), and find a 'triality' automorphism of the quotient group $\mathrm{O}_8^+(3)$. This automorphism is given (see [18]) in terms of the 'standard generators' a and b by the map $(a, b) \mapsto ((ab^2)^7, ((abab^4ab^4)^4)^{ab^4ab^4})$. By applying this map to the generators of the double cover we obtain two more 8-dimensional representations of $2^{2\cdot}\mathrm{O}_8^+(3)$, and their direct sum is invariant under all the automorphisms of this group. We now adjoin all these automorphisms using the usual 'standard basis' technique, see for example [11]. Finally, an isotropic 1-space in the original 8-dimensional representation has 3360 images in the 24-space under the action of the group, and this gives rise to the required permutation representation of $\mathrm{O}_8^+(3){:}\mathrm{S}_4$.

To make $\mathrm{S}_4(4){:}4$, we take the direct sum of a natural representation of $\mathrm{S}_4(4)$ over GF(4) with its field automorph, and adjoin a field automorphism to obtain an 8-dimensional representation of $\mathrm{S}_4(4){:}2$. This can now be written over GF(2). Again, we apply the outer automorphism of $\mathrm{S}_4(4){:}2$ to get another 8-dimensional representation. Taking the direct sum of these two, we may adjoin the outer automorphism in the usual way. We now permute a suitable orbit of 510 vectors. But these vectors form 170 subspaces of dimension 2, so we use Magma to find the corresponding block system and write the group as permutations on 170 points.

The structures of the two remaining local subgroups in the list

$$
\begin{aligned}
(5^2{:}(4 \circ \mathrm{Q}_8) \times \mathrm{U}_3(5)){:}\mathrm{S}_3 &\cong (5^2{:}4\mathrm{S}_4 \times \mathrm{U}_3(5){:}\mathrm{S}_3)\frac{1}{\mathrm{S}_3}\\
(7^2{:}(3 \times 2\mathrm{A}_4) \times \mathrm{L}_2(7)).2 &\cong (7^2{:}(3 \times 2^{\cdot}\mathrm{S}_4^-) \times \mathrm{L}_2(7){:}2)\frac{1}{2}
\end{aligned}
$$

are given in [17]. Here we need to construct the affine groups $5^2{:}4\mathrm{S}_4$ inside $5^2{:}\mathrm{GL}_2(5)$ and $7^2{:}(3 \times 2^{\cdot}\mathrm{S}_4^-)$ inside $7^2{:}\mathrm{GL}_2(7)$. We describe these constructions in Section 5 below. We end up with permutation representations on $5^2 + 126 = 151$ points and $7^2 + 8 = 57$ points respectively.

4 Subwreath products

Next we consider the four cases which we have called subwreath products:

$$\mathrm{A_6}^{3\cdot}(2 \times \mathrm{S_4}) \cong (\mathrm{S_6} \wr \mathrm{S_3})\frac{1}{2}.2$$
$$(\mathrm{A_7} \times (\mathrm{A_5} \times \mathrm{A_5}){:}2^2){:}2 \cong (\mathrm{S_7} \times (\mathrm{S_5} \wr 2))\frac{1}{2}$$
$$\mathrm{S_5}^3{:}\mathrm{S_3} \cong \mathrm{S_5} \wr \mathrm{S_3}$$
$$\mathrm{L_2}(11)^2{:}4 \cong (\mathrm{L_2}(11){:}2 \wr 2)\frac{1}{2}$$

The precise structures of these groups are rather subtle, so we shall justify them here. First note that $(\mathrm{A_5} \times \mathrm{A_{12}}){:}2$ contains a subgroup $(\mathrm{A_5} \times (\mathrm{A_5} \times \mathrm{A_7}){:}2){:}2$. According to [13] these two subgroups $\mathrm{A_5}$ are conjugate in the Monster, so the normaliser of this $\mathrm{A_5} \times \mathrm{A_5}$ is twice as big. Moreover, inside the $7A$-centraliser $7 \times \mathrm{He}$ we see $(\mathrm{A_5} \times \mathrm{A_5}){:}2^2 < \mathrm{S_4}(4){:}2$, which gives the precise structure of the group as the unique subgroup of shape $((\mathrm{A_5} \times \mathrm{A_5}){:}2^2 \times \mathrm{A_7}){:}2$ in $\mathrm{S_5} \wr 2 \times \mathrm{S_7}$. The subgroup $(\mathrm{A_5} \times \mathrm{A_5}){:}2^2$ commuting with $\mathrm{A_7}$ contains a wreathing involution and an involution extending both $\mathrm{A_5}$ factors to $\mathrm{S_5}$. Note that the ATLAS [4] gives an incorrect structure for this group.

In $(\mathrm{A_5} \times \mathrm{A_{12}}){:}2$ pick a subgroup $\mathrm{A_5}^3$ such that the $\mathrm{A_5} \times \mathrm{A_5}$ in $\mathrm{A_{12}}$ fixes two points. This $\mathrm{A_5}^3$ has trivial $\mathbb{M}$-centraliser and has normaliser $\mathrm{S_5} \times \mathrm{S_5} \wr 2$ in $\mathrm{S_5} \times \mathrm{S_{10}} < (\mathrm{A_5} \times \mathrm{A_{12}}){:}2$. We can choose notation so that $\mathrm{A_5}^3$ is a subgroup of $((\mathrm{A_5} \times \mathrm{A_5}){:}2^2 \times \mathrm{A_7}){:}2$, and its normaliser therein is then $\mathrm{S_5} \wr 2 \times \mathrm{S_5}$. In the two groups $\mathrm{S_5} \times \mathrm{S_5} \wr 2$ different pairs of $\mathrm{A_5}$s are swapped. Therefore the full $2 \wr \mathrm{S_3} \cong 2 \times \mathrm{S_4}$ of outer automorphisms of $\mathrm{A_5}^3$ is realised in its Monster normaliser, and we get $\mathrm{N}_{\mathbb{M}}(\mathrm{A_5}^3) \cong \mathrm{S_5} \wr \mathrm{S_3}$.

The trickiest case to describe is the normaliser of $\mathrm{A_6}^3$. The actual structure is as follows: Note that $\mathrm{S_6} \wr \mathrm{S_3}$ has a normal subgroup $\mathrm{A_6}^3$ with quotient $2 \wr \mathrm{S_3} \cong 2 \times \mathrm{S_4}$, so there are two subgroups of shape $\mathrm{A_6}^3{:}\mathrm{S_4}$. We take the one which contains $\mathrm{A_6} \wr \mathrm{S_3}$, and adjoin an automorphism which extends all three $\mathrm{A_6}$ factors to $\mathrm{M_{10}}$.

Norton [13] notes that the $\mathrm{A_5}$ which centralises $\mathrm{A_{12}}$ extends uniquely to a conjugacy class of $\mathrm{A_6}$. Moreover, such an $\mathrm{A_6}$ is the centraliser in $\mathbb{M}$ of the subgroup $(\mathrm{A_6} \times \mathrm{A_6}){:}2^2$ of $\mathrm{A_{12}}$ consisting of the even permutations in $\mathrm{S_6} \wr 2$. Therefore all such $\mathrm{A_6}$s are conjugate in the normaliser of the $\mathrm{A_5}$. Also, both classes of $\mathrm{A_5}$ in the $\mathrm{A_6}$ are of this type, so the $\mathrm{A_6}$-normaliser contains an element interchanging these two classes. Since the factors of this $\mathrm{A_6} \times \mathrm{A_6}$ also contain $\mathrm{A_5}$s of the same type, it follows that the full normaliser of this $\mathrm{A_6} \times \mathrm{A_6} \times \mathrm{A_6}$ is transitive on the three factors. Hence it has order $48.|\mathrm{A_6}|^3$.

For the sake of definiteness we define the 5-point and 6-point $\mathrm{A_5}$s in the $\mathrm{A_6}$ factors as follows. Pick an $\mathrm{A_5}$ in the first $\mathrm{A_6}$ and define it arbitrarily to be a 5-point $\mathrm{A_5}$. Its centraliser is $\mathrm{A_{12}}$, containing the other two $\mathrm{A_6}$ factors. We define the 5-point $\mathrm{A_5}$s to be the ones which fix 7 of the 12 points on which the $\mathrm{A_{12}}$ acts. Inside $\mathrm{A_{12}}$ we already see an involution centralising the first $\mathrm{A_6}$ and swapping the other two in such a way that the 5-point $\mathrm{A_5}$s get swapped also. We have seen in our investigation of $(\mathrm{A_7} \times (\mathrm{A_5} \times \mathrm{A_5}){:}2^2){:}2$ that there is an involution centralising the $\mathrm{A_7}$ (and therefore centralising one of the other $\mathrm{A_6}$ factors in our $\mathrm{A_6}^3$), swapping the $\mathrm{A_5}$ factors. But in our notation these two $\mathrm{A_5}$s are both 5-point $\mathrm{A_5}$s in their respective $\mathrm{A_6}$s. Hence the normaliser of $\mathrm{A_6}^3$ contains $\mathrm{A_6} \wr \mathrm{S_3}$.

Moreover, the subgroup $(\mathrm{S}_5 \times \mathrm{S}_6 \times \mathrm{S}_6)\frac{1}{2}$ inside $(\mathrm{S}_5 \times \mathrm{S}_{12})\frac{1}{2}$ extends this to a well-defined group ${\mathrm{A}_6}^3{:}\mathrm{S}_4$, namely the unique subgroup of index 2 in $\mathrm{S}_6 \wr \mathrm{S}_3$ which contains $\mathrm{A}_6 \wr \mathrm{S}_3$. To obtain the full normaliser we now just need to adjoin an element swapping the 5-point A_5s with the 6-point ones. There are essentially two possibilities for such an element: either it extends all the A_6 factors to $\mathrm{PGL}_2(9)$, or it extends tham all to M_{10}. We need to determine which exists in the Monster.

Take a D_{10} in one of the factors: its centraliser in the Monster is a copy of the Harada–Norton group. It was discovered in the course of the original construction of the latter group that it contains a subgroup $(\mathrm{A}_6 \times \mathrm{A}_6){}^{\cdot}\mathrm{D}_8$ in which there is an element which extends one factor to $\mathrm{PGL}_2(9)$ and the other to M_{10}. Since this does not centralise the remaining A_6 factor of ${\mathrm{A}_6}^3$, but does centralise the D_{10}, it must extend it to $\mathrm{PGL}_2(9)$. But we can now multiply this element by an element extending two factors to S_6, to get an element which extends all three to M_{10}.

In the last case, we see two non-conjugate subgroups $(\mathrm{L}_2(11) \times \mathrm{L}_2(11)){:}2$ inside $(\mathrm{L}_2(11) \times \mathrm{M}_{12}){:}2$, coming from the two classes of $\mathrm{L}_2(11)$ in M_{12}. If we take the maximal $\mathrm{L}_2(11)$ in M_{12}, then Norton's classification of subgroups containing $5A$-elements shows that the two factors are conjugate in the Monster. Therefore the maximal subgroup we want is a subgroup of index 2 in $(\mathrm{L}_2(11){:}2) \wr 2$. But the 11-centraliser in $\mathbb{M}$ is $11 \times \mathrm{M}_{12}$, which does not contain $11 \times \mathrm{D}_{22}$, whereas both subgroups of shape $\mathrm{L}_2(11)^2{:}2^2$ do. It follows that our group has the shape $\mathrm{L}_2(11)^2{:}4$.

5 Local subgroups of affine type

We divide the 'easy' local subgroups into two types: the 'affine' type which are normalisers of elementary abelian subgroups, and the 'extraspecial' type, which are the normalisers of extraspecial groups. In this section we treat the local maximal subgroups which are of affine type. These are $5^4{:}(3 \times 2{}^{\cdot}\mathrm{L}_2(25)){:}2_2$, $7^2{:}2{}^{\cdot}\mathrm{L}_2(7)$, $11^2{:}(5 \times 2{}^{\cdot}\mathrm{A}_5)$, $13^2{:}2{}^{\cdot}\mathrm{L}_2(13){:}4$, and $41{:}40$. We also need to construct certain affine groups as part of the construction of the subdirect products described above. These groups are $3^2{:}2{}^{\cdot}\mathrm{S}_4^+$, $5^2{:}4\mathrm{S}_4$ and $7^2{:}(3 \times 2{}^{\cdot}\mathrm{S}_4^-)$.

In fact, affine groups are very easy to construct once we have the action of the complement on the vector space. Thus we need to construct the 2-dimensional (faithful) representations of $2{}^{\cdot}\mathrm{S}_4^+ \cong \mathrm{GL}_2(3)$ over $\mathrm{GF}(3)$, of $4\mathrm{S}_4$ over $\mathrm{GF}(5)$, of $3 \times 2{}^{\cdot}\mathrm{S}_4^-$ and $\mathrm{SL}_2(7)$ over $\mathrm{GF}(7)$, of $5 \times 2{}^{\cdot}\mathrm{A}_5$ over $\mathrm{GF}(11)$, of $2{}^{\cdot}\mathrm{L}_2(13){:}4 = \mathrm{GL}_2(13)\frac{1}{3}$ over $\mathrm{GF}(13)$, and of $3 \times \mathrm{SL}_2(25)$ over $\mathrm{GF}(25)$. All these are easy to make. Finally we need to write the last representation over $\mathrm{GF}(5)$, and adjoin an element realising the field automorphism of $\mathrm{GF}(25)$.

The group $4\mathrm{S}_4$ mentioned here is the unique subgroup of index 5 in $\mathrm{GL}_2(5)$; it has shape $2{}^{\cdot}(\mathrm{A}_4 \times 2){}^{\cdot}2 \cong 2{}^{\cdot}\mathrm{A}_4{:}4 \cong 4{}^{\cdot}\mathrm{A}_4.2$ and has centre of order 4.

The precise structure of the group $5^4{:}(3 \times 2{}^{\cdot}\mathrm{L}_2(25)){:}2_2$ is rather subtle, and a proof does not appear to be readily available in the literature, so we sketch a proof here. Calculations which were suppressed in [17] and revealed in [16] show that the Monster contains exactly four conjugacy classes of elementary abelian groups of order 5^4, just one of which is not conjugate to a subgroup of the normal subgroup 5_+^{1+6} of the maximal subgroup $5_+^{1+6}{:}2{}^{\cdot}\mathrm{J}_2{:}4$. A 5^4 of this last class is of pure $5B$-type, and by symmetry its normaliser is transitive on its non-trivial elements.

Any given 5-element in the 5^4 is contained in a unique 5^2 of type $5B_6(i)$, and the remaining 5^2 subgroups are equally divided between those of type $5B_6(ii)$ and $5B_6(iii)$, since the Sylow 5-subgroup of J_2 contains equal numbers of 5-elements in each conjugacy class. Therefore each 5-element is in exactly 15 of each of these two types of 5^2 inside 5^4. Therefore the numbers of 5^2-subgroups of the three types in 5^4 are 26, 390 and 390 respectively.

Now the calculations of [17] can be extended to show that the intersection of $\mathrm{N}(5B)$ and $\mathrm{N}(5B_6(ii))$ has order $5^5.2^3$. This is because the normaliser in $2^{\cdot}\mathrm{J}_2{:}4$ of a cyclic group X of J_2-type $5AB$ is $(\mathrm{D}_{10} \times \mathrm{GL}_2(5)).2$ which is transitive on the 24 non-trivial cosets of the commutator of X on the 5^6. [Notice incidentally that the resulting group of order 8 acts faithfully on the $5B_6(ii)$ subgroup, and hence the centralizer of the $5B_6(ii)$ is only the 5^4, not $5^4{:}2$ as stated in [17]. This does not affect the results of [17], however.]

It follows from this that $\mathrm{N}(5B_6(ii))$ has order $2^4.3.5^5$, and then from the transitivity of $\mathrm{N}(5^4)$ on the 390 subgroups of type $5B_6(ii)$ it contains, that $\mathrm{N}(5^4)$ has order $2^5.3^3.5^6.13$.

Obviously $\mathrm{N}(5^4)$ is transitive on the 26 subgroups of type $5B_6(i)$ in the 5^4. Indeed, it is not hard to see that the elements of order 5 outside the 5^4 fix only one of these subgroups, and therefore our group acts 2-transitively on these 26 'points'. In particular, its action contains $\mathrm{L}_2(25)$ as a normal subgroup. Therefore $\mathrm{N}(5^4)$ has a normal subgroup $5^4{:}\mathrm{SL}_2(25)$ (of index 6). (Note that the normaliser of $\Omega_4^-(5)$ in $\mathrm{GL}_4(5)$ does not act transitively on the nonzero vectors of $\mathbb{F}_5^4$, so that $\mathrm{N}(5^4)$ does not contain a normal subgroup $5^4.\Omega_4^-(5)$.) Since $\mathrm{SL}_2(25)$ has no outer automorphism of order 3, our group actually contains $5^4{:}(3 \times \mathrm{SL}_2(25))$ to index 2.

Now Fi_{24}' does not contain $\mathrm{SL}_2(25)$ (although it does contain $\mathrm{L}_2(25)$), so the complementary group of shape $(3 \times \mathrm{SL}_2(25)).2$ is contained in $6^{\cdot}\mathrm{Suz}{:}2$. In particular, the outer half of the group consists of elements which realise a non-trivial outer automorphism of $\mathrm{SL}_2(25)$ while not fusing the two classes of 5-elements. Therefore it is a field automorphism, and therefore it inverts the scalars of order 3. This determines the precise structure of the group, which we write as $5^4{:}(3 \times \mathrm{SL}_2(25)){:}2_2$ in conformity with Atlas notation.

6 Local subgroups of extraspecial type

Here we treat the groups $5_+^{1+6}{:}2^{\cdot}\mathrm{J}_2{:}4$, $7_+^{1+4}{:}(3 \times 2^{\cdot}\mathrm{S}_7^-)$, $13_+^{1+2}{:}(3 \times 4\mathrm{S}_4)$, as well as the group $3_+^{1+12}{:}2^{\cdot}\mathrm{Suz}{:}2$, which we use in Section 11. All of these groups are of extraspecial type, of shape $p_+^{1+2n}{:}H$, and are subgroups of groups of shape $p_+^{1+2n}{:}\mathrm{Sp}_{2n}(p).\mathrm{C}_{p-1}$. In each case, there is a unique group of the given shape, since the complement acts absolutely irreducibly on the p^{2n} (ensuring that there is a unique embedding of H into $\mathrm{Sp}_{2n}(p).\mathrm{C}_{p-1}$), and contains the central involution of $\mathrm{Sp}_{2n}(p)$ (ensuring that $p^{2n}{:}H$ has a unique class of complements). These can all be constructed by the method described in [11] for constructing $3_+^{1+12}{:}2^{\cdot}\mathrm{Suz}{:}2$. We start with the 6-dimensional representation of $2^{\cdot}\mathrm{J}_2{:}4$ over GF(5), the 4-dimensional representation of $3 \times 2^{\cdot}\mathrm{S}_7^-$ over GF(7), the 2-dimensional representation of $3 \times 4\mathrm{S}_4$ over GF(13) and the 12-dimensional representation of $2^{\cdot}\mathrm{Suz}{:}2$ over GF(3), and follow the recipe.

Unfortunately there are some (minor) errors in [11], so we describe the method again here. To construct a representation of a group $G \cong p_+^{1+2n}{:}H$ where H is a subgroup of the general symplectic group $\mathrm{Sp}_{2n}(p){:}\mathrm{C}_{p-1}$, we take a representation over $\mathrm{GF}(p)$ of the affine group $G/C_p \cong$

$p^{2n}{:}H$, of shape $M \cdot 1$ where M is the natural module for H of dimension $2n$ over $\mathrm{GF}(p)$. The dual of $M \cdot 1$ has shape $1 \cdot (M \otimes 1^{\bar{\lambda}})$ for some 1-dimensional module $1^{\bar{\lambda}}$, so by tensoring with 1^{λ} we obtain a module of shape $1^{\lambda} \cdot M$. We then glue them together to make a module of shape $1^{\lambda} \cdot M \cdot 1$.

This gluing process leaves just one entry of the matrices undetermined: allowing this entry to take all possible values in $\mathrm{GF}(p)$ gives us a group $\tilde{G}$ with a normal subgroup of order p, modulo which we recover the affine group.

We illustrate the method by constructing $13^{1+2}_{+}{:}(3 \times 4\mathrm{S}_4)$. First, since $H \cong 3 \times 4\mathrm{S}_4$ is unique up to conjugacy in $\mathrm{GL}_2(13)$, we may take it to be generated by the matrices $\begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ over $\mathrm{GF}(13)$. In this case the scalar by which the matrix A multiplies the symplectic form is given by $\det A$, so we extend to a 4-dimensional representation of H generated by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now the symplectic form may be taken as $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the extraspecial subgroup consists of all matrices of the form

$$B(\lambda, v) := \begin{pmatrix} 1 & 0 & 0 \\ Sv^{\top} & \mathrm{I}_2 & 0 \\ \lambda & v & 1 \end{pmatrix}$$

where v is a 2-dimensional row vector and λ is a scalar. Notice that the commutator of $B(\lambda, v)$ with $B(\mu, w)$ is $B(2vSw^{\top}, 0)$, which shows that this group is extraspecial whenever p is odd. It also explains why the construction does not work when $p = 2$.

An analogous construction works in the other cases, except that the scalar which multiplies the symplectic form can no longer be defined as the determinant of the matrix. We obtain matrix representations of dimension 8, 6, 4 and 12 respectively over $\mathrm{GF}(5)$, $\mathrm{GF}(7)$, $\mathrm{GF}(13)$ and $\mathrm{GF}(3)$.

We can then obtain permutation representations on the p^{1+2n} cosets of a complement by permuting an orbit of 1-spaces, consisting of all the 1-spaces which are not in the invariant subspace of codimension 1. The permutation representations have degrees $5^7 = 78125$, $7^5 = 16807$, $13^3 = 2197$ and $3^{13} = 1594323$ respectively. (We have not actually made the last representation.)

For groups of these shapes (and for the more complicated groups described below) there are no faithful primitive permutation representations, but the following concept of pseudo-primitive actions captures the idea of being as close as possible to primitive for faithful representations. Essentially, we want that if G acts on any non-trivial block system then the action is not faithful. Equivalently, that the point stabiliser is maximal subject to containing no minimal normal subgroup of G.

Definition 1. A group G is said to act *pseudo-primitively* on a set Ω if:

1. G acts transitively on Ω;
2. $|\Omega| \geqslant 2$;
3. for any/all $\omega \in \Omega$ and for all $H \leqslant G$ such that $G_\omega < H$ we have $\mathrm{Core}_G(H) > \mathrm{Core}_G(G_\omega)$.

With this definition, the permutation representations we have just constructed are pseudo-primitive.

7 Hard local subgroups

Some of these were constructed from scratch utilising knowledge of the local structure of the Monster. Others were constructed by finding an explicit copy of the desired maximal subgroup inside the Monster as constructed by computer, and finding a faithful permutation representation of the maximal subgroup on a suitable orbit of vectors. The groups constructed from scratch were $7^2.7.7^2{:}\mathrm{GL}_2(7)$, $5^{3\cdot}5^{3\cdot}(2 \times \mathrm{L}_3(5))$, $5^2.5^2.5^4{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$, $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$, and $3_+^{1+12\cdot}2^{\cdot}\mathrm{Suz}{:}2$. The groups found as explicit subgroups of the Monster were $3^2.3^5.3^{10\cdot}(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$ and $3^3.3^2.3^6.3^6{:}(\mathrm{L}_3(3) \times \mathrm{SD}_{16})$. As each of these constructions exhibits its own difficulties, we treat each one in a separate section. Note that alternative constructions of the three groups $7^2.7.7^2{:}\mathrm{GL}_2(7)$, $5^2.5^2.5^4{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$ and $3^2.3^5.3^{10\cdot}(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$ are given by Richardson [15].

8 The subgroup $7^{2+1+2}{:}\mathrm{GL}_2(7)$

We shall show that the subgroup $H \cong 7^{2+1+2}{:}\mathrm{GL}_2(7)$ of $\mathbb{M}$ is isomorphic to a maximal parabolic subgroup $P \cong 7^{2+1+2}{:}\mathrm{GL}_2(7)$ of the Chevalley group $\mathrm{G}_2(7)$. The group P is the stabiliser of an isotropic point in the natural representation of $\mathrm{G}_2(7)$. (In the Magma incarnation of $\mathrm{G}_2(7)$ all basis vectors except the fourth are isotropic.)

Now $K = \mathrm{N}_{\mathbb{M}}(7B) \cong 7_+^{1+4}{:}(3 \times 2^{\cdot}\mathrm{S}_7^-)$ contains a Sylow 7-subgroup, S say, of $\mathbb{M}$. We easily calculate that $\mathrm{Z}(S)$ has order 7 and that $\mathrm{Z}_2(S)$ has order 7^2; also $K = \mathrm{N}_{\mathbb{M}}(\mathrm{Z}(S))$. Thus, since non-trivial normal subgroups of p-groups intersect the centre non-trivially, $\mathrm{Z}_2(S)$ is the unique normal subgroup of S of order 7^2. So the group $H \cong 7^{2+1+2}{:}\mathrm{GL}_2(7)$ is $\mathrm{N}_{\mathbb{M}}(\mathrm{Z}_2(S))$, and we obtain $\mathrm{O}_7(H) = \mathrm{C}_S(\mathrm{Z}_2(S))$. Now S has exponent 7, and so repeated application of Gaschütz's Theorem shows that H is a split extension of 7^{2+1+2} by $\mathrm{GL}_2(7)$. We convert $\mathrm{O}_7(H) \cong 7^{2+1+2}$ and $\mathrm{O}_7(P) \cong 7^{2+1+2}$ into PC-groups in Magma. We then apply the Magma command StandardPresentation to each of these 7^{2+1+2}'s and find that they are isomorphic. Applying AutomorphismGroup to the PC-group 7^{2+1+2} we find that $|\mathrm{Aut}\, 7^{2+1+2}| = 7^6.|\mathrm{GL}_2(7)|$, and so $|\mathrm{Out}\, 7^{2+1+2}| = 7^3.|\mathrm{GL}_2(7)|$. Furthermore, $\mathrm{Aut}\, 7^{2+1+2}$ has a normal subgroup of order 7^6 of automorphisms of 7^{2+1+2} that centralise the central 7^2 thereof. Now in $\mathrm{G}_2(7)$ we see a subgroup $\mathrm{GL}_2(7)$ of $\mathrm{Aut}\, 7^{2+1+2}$. But (among characteristic 7 irreducibles) only the 5-dimensional irreducible module of $\mathrm{SL}_2(7)$ has nonzero 1-cohomology, and so $\mathrm{Out}\, 7^{2+1+2}$ contains just one conjugacy class of subgroups $\mathrm{GL}_2(7)$. Since $\mathrm{Inn}\, 7^{2+1+2} \cong 7_+^{1+2}$, we see that $\mathrm{Aut}\, 7^{2+1+2}$ contains a unique class of $\mathrm{GL}_2(7)$. Hence there is a unique group of shape $7^{2+1+2}{:}\mathrm{GL}_2(7)$, as required.

The group $H \cong 7^{2+1+2}{:}\mathrm{GL}_2(7)$ permutes the $7^4 = 2401$ involutions of the normal subgroup $7^{2+1+2}{:}2$ transitively and faithfully. Each such involution has centraliser $7{:}\mathrm{GL}_2(7) \cong (7 \times \mathrm{SL}_2(7)){:}6$, and these define the 2401 complements $\mathrm{GL}_2(7)$. We used the Subgroups facility in Magma in order to search for faithful permutation representations of smaller degree. Thus we found a faithful permutation representation of H of degree 392 (which is the unique faithful permutation representation of degree at most 392). This representation is on the cosets of the subgroup $\mathrm{N}_H(Q) \cong (7_+^{1+2} \times 7){:}6^2$ where Q is a Sylow 7-subgroup of a complementary $\mathrm{GL}_2(7)$. The restriction of the natural $\mathrm{G}_2(7)$ module gives a faithful representation of H of dimension 7 over GF(7). The 6-dimensional submodule and 6-dimensional quotient of the restriction also represent H faithfully.

Note that the upper and lower central series of $\mathrm{O}_7(H) \cong 7^{2+1+2}$ coincide. The terms of these series have orders 1, 7^2, 7^3 and 7^5. The term of order 7^3 is elementary abelian. The complementary $\mathrm{GL}_2(7)$ has a kernel of order 3 in its action on $\mathrm{Z}(\mathrm{O}_7(H))$.

9 The group $5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$

Both $5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$ and $5_+^{1+6}{:}2\mathrm{J}_2.4$ contain a Sylow 5-subgroup, S say, of $\mathbb{M}$. We have already made a copy of $K := \mathrm{N}_{\mathbb{M}}(\mathrm{Z}(S)) \cong 5_+^{1+6}{:}2\mathrm{J}_2.4$ and are trying to construct $H := \mathrm{N}_{\mathbb{M}}(\mathrm{Z}_2(S)) \cong 5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$. We shall need to consider the subgroups $T := \mathrm{C}_{\mathbb{M}}(\mathrm{Z}_2(S)) \cong 5^{2+2+4}{:}\mathrm{S}_3$ and $J := \mathrm{N}_{\mathbb{M}}(S) \cong S{:}(\mathrm{S}_3 \times 4 \times 4)$. We have that $T < J$ and J is maximal in both H and K.

Note that H is a split extension because any element of order 3 in T centralises only the normal subgroup $\mathrm{Z}_2(S) \cong 5^2$ of $\mathrm{O}_5(H) \cong 5^{2+2+4}$. Thus by a Frattini argument its centraliser and normaliser in H are $(3 \times 5^2).\mathrm{GL}_2(5)$ and $(\mathrm{S}_3 \times 5^2).\mathrm{GL}_2(5)$. These are necessarily the split extensions $3 \times 5^2{:}\mathrm{GL}_2(5)$ and $\mathrm{S}_3 \times 5^2{:}\mathrm{GL}_2(5)$.

Note that $\mathrm{Z}(S)$ and $\mathrm{Z}_2(S)$ are the unique normal subgroups of S of orders 5 and 5^2 respectively. The group K is available to us as permutations on 78125 points.

Inside $K \cong 5_+^{1+6}{:}2\mathrm{J}_2.4$, we locate a copy of $J = \mathrm{N}_{\mathbb{M}}(S) \cong 5^{2+2+4}{:}(\mathrm{S}_3 \times 4 \times 5{:}4)$, and inside J we locate the copy of $T = \mathrm{C}_{\mathbb{M}}(\mathrm{Z}_2(S)) \cong 5^{2+2+4}{:}\mathrm{S}_3$. Note that we have $S = \mathrm{O}_5(J)$ and $T = \mathrm{C}_J(\mathrm{Z}_2(S)) = \mathrm{C}_J(\mathrm{Z}_2(\mathrm{O}_5(J)))$, so that S and T can be obtained from J by standard group theoretic operations. Thus J and T can easily be obtained as permutation groups.

In Magma, we convert T into a PC-group and calculate its automorphism group, which has order $5^6.6.480$. So $\mathrm{Aut}\,T \cong \mathrm{Inn}\,T.\mathrm{GL}_2(5) \cong 5^{2+4}{:}\mathrm{S}_3.\mathrm{GL}_2(5)$. Now the action of $\mathrm{SL}_2(5)$ on 5^{2+2+4} has only 1- and 2-dimensional constituents, both of which have zero 1-cohomology. Thus there is a unique way to extend $5^{2+2+4}{:}\mathrm{S}_3$ to $5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$.

To construct H as a permutation group, we first identify a suitable point stabiliser as the centraliser of the central involution of $\mathrm{GL}_2(5)$. This group is core-free and has shape $(5^2{:}\mathrm{S}_3 \times \mathrm{SL}_2(5)){:}4$ and index $5^6 = 15625$.

So far we have a maximal subgroup $J \cong 5^{2+2+4}{:}(\mathrm{S}_3 \times 4 \times 5{:}4)$ of H, acting as permutations on 78125 points, and our next task is to find the appropriate representation of J on 15625 points. The point stabiliser in J is a group of shape $5^2{:}(\mathrm{S}_3 \times 4 \times 5{:}4)$, being the centraliser of a suitable involution. This involution may be identified by observing that it acts as -1 on $\mathrm{Z}_2(S)$, and centralises $5^2{:}\mathrm{S}_3$. Thus we obtain J acting on 15625 points, using the MAGMA command CosetImage.

We obtain explicitly the subgroup $T = \mathrm{C}_J(\mathrm{Z}_2(S)) = \mathrm{C}_J(\mathrm{Z}_2(\mathrm{O}_5(J)))$ of J, and try to extend its normaliser. We find the automorphism group of T and apply some automorphism to a generating set for T. A standard-basis algorithm produces a permutation realising this automorphism. But since the point stabiliser is self-normalising in T, it follows that T contains its own centraliser in S_{15625}. Therefore all such permutations are in $\mathrm{N}_{\mathrm{S}_{15625}}(T) \cong 5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$, which is the group we want.

We obtain a faithful permutation representation of H of degree 750 on the cosets of the normaliser of a Sylow 5-subgroup of $\mathrm{GL}_2(5)$. The group $\mathrm{O}_5(H) \cong 5^{2+2+4}$ is an exponent 5 group with coincident upper and lower central series. The terms of these series have orders 1, 5^2, 5^4 and 5^8, and the term of order 5^4 is abelian. The actions of $\mathrm{S}_3 \times \mathrm{GL}_2(5)$ on the factors of the upper central series (5^2, 5^2, 5^4) are $V \otimes W$ where V is the S_3 module 1 (trivial), 2, 2 and W is the $\mathrm{GL}_2(5)$ module $U \otimes 1^i$, 1^i, U respectively. We have used U to denote a natural 2-dimensional module of $\mathrm{GL}_2(5)$, and 1^i denotes a 1-dimensional module of $\mathrm{GL}_2(5)$ that is faithful for $\mathrm{GL}_2(5)/\mathrm{SL}_2(5) \cong \mathrm{C}_4$.

10 The group $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$

First we describe the structure of this group precisely. Inside $3^{1+12\cdot}2^{\cdot}\mathrm{Suz}{:}2$ we see the structure $3^7.3^6.6_2{}^{\cdot}\mathrm{U}_4(3)^{\cdot}(2^2)_{133}$ which shows that even a subgroup $2^{\cdot}\mathrm{U}_4(3)$ fails to split off the normal 3^8. The automorphism of $\mathrm{O}_8^-(3)$ cannot be the automorphism 2_2, as there are no elements of order 82 in $\mathbb{M}$ (see [4]); this reason also rules out the possibility of this group being $3^8.(\mathrm{O}_8^-(3) \times 2)$. Nor can it be the automorphism 2_1, as if so there would be a reflection centralising $3^7.\mathrm{O}_7(3)$, but neither the Baby Monster nor the largest Conway group Co_1 contains such a subgroup. Therefore we must have the 2_3 automorphism. Note that the extension of $\mathrm{O}_8^-(3)$ by its 2_3 automorphism is necessarily non-split. There are two absolutely irreducible 8-dimensional GF(3)-modules for $\mathrm{O}_8^-(3)^{\cdot}2_3$. These two modules can be obtained from each other by tensoring with a linear representation, and are also conjugate under an outer automorphism of $\mathrm{O}_8^-(3)^{\cdot}2_3$. Using MAGMA 2.10, we find these 8-dimensional modules each have 1-dimensional 2-cohomology for $\mathrm{O}_8^-(3)^{\cdot}2_3$. Therefore, there is a unique group of type $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$.

Remark. The natural module of $\mathrm{O}_8^-(3)$ has 2-dimensional 2-cohomology, which can be calculated using MAGMA [3]. This leads to two isomorphism classes of $3^{8\cdot}\mathrm{O}_8^-(3)$. It can be shown using Clifford Theory that these groups have different character degrees, both differing from those of $3^8{:}\mathrm{O}_8^-(3)$. The non-Monster $3^{8\cdot}\mathrm{O}_8^-(3)$ has a faithful permutation representation of degree 3321 on the cosets of a subgroup $3^{7\cdot}\mathrm{O}_7(3){:}2$, and this extends to a permutation representation of $3^{8\cdot}\mathrm{O}_8^-(3){:}2_1$ on the cosets of $2 \times 3^{7\cdot}\mathrm{O}_7(3){:}2$.

Our next task is to construct a 3-modular representation of $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$ by pasting together representations of $\mathrm{O}_8^-(3)^{\cdot}2_3$ in such a way that the resulting representation is forced to be a representation of $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$. Specifically, our representation will take the form $8 \cdot X \cdot 1$ where $8 \cdot X$ and $X \cdot 1$ are uniserial modules for $\mathrm{O}_8^-(3)^{\cdot}2_3$. If we can find a suitable module X such that

$O_8^-(3){}^{\cdot}2_3$ does not have a uniserial module of shape $8 \cdot X \cdot 1$, then we shall have constructed a group of shape $3^{8\cdot}O_8^-(3){}^{\cdot}2_3$ as required.

If $O_8^-(3){}^{\cdot}2_3$ is generated by $g_1, g_2, \ldots$, then g_i can be represented by the matrices (written over GF(3))

$$\begin{bmatrix} A_i & 0 \\ B_i & C_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_i & 0 \\ D_i & E_i \end{bmatrix}$$

in representations $8^+ \cdot X$ and $X \cdot 1^+$ respectively. (Of course, E_i is the 1×1 identity matrix.) Pasting these representations together gives matrices

$$h_i = \begin{bmatrix} A_i & 0 & 0 \\ B_i & C_i & 0 \\ F_i & D_i & E_i \end{bmatrix}$$

where F_i is an as yet undetermined 1×8 matrix. Thus the h_i generate a subgroup $W.O_8^-(3){}^{\cdot}2_3$ of $V.O_8^-(3){}^{\cdot}2_3$ where here V is the natural module $8^+ \otimes (1^+)^* \cong 8^+$ and W is a GF(3)-submodule of V. The natural map of quotienting by W maps h_i to g_i. In our case the matrices h_i generate either $O_8^-(3){}^{\cdot}2_3$, $3^8{:}O_8^-(3){}^{\cdot}2_3$ or $3^{8\cdot}O_8^-(3){}^{\cdot}2_3$. In the case when the h_i generate $3^8{:}O_8^-(3){}^{\cdot}2_3$ we can adjust the F_i and force the h_i to generate $O_8^-(3){}^{\cdot}2_3$; if no such adjustment on the F_i is possible then the h_i generate $3^{8\cdot}O_8^-(3){}^{\cdot}2_3$.

We found that the permutation module over GF(3) of $O_8^-(3){}^{\cdot}2_3$ on 1066 points has structure $1^+ \oplus Y$ where Y has the following structure:

$$\begin{array}{c}
35^+ \\
195^+ \\
\hline
1^- \oplus 8^+ \oplus 8^- \oplus \begin{array}{c} 35^- \\ 518^{\varepsilon} \\ 35^- \end{array} \\
\hline
195^+ \\
35^+
\end{array}$$

In particular, the module 195^+ can be glued to both 1^- and 8^-, so this looks like a good candidate to try.

Now we can cut out $195^+ \cdot 1^-$ and $8^- \cdot 195^+$ from this representation and tensor with 1^- to get $195^- \cdot 1^+$ and $8^+ \cdot 195^-$. We can then use a standard basis algorithm to ensure that both copies of 195^- are represented by the same matrices, and glue the representations together as described above. Thus we obtain a representation of shape $8^+ \cdot 195^- \cdot 1^+$ for a group of shape $N.O_8^-(3){}^{\cdot}2_3$, where N is either trivial or 3^8.

If this group has a subgroup $O_8^-(3){}^{\cdot}2_3$, then the corresponding matrices h_i have to satisfy the relations of $O_8^-(3){}^{\cdot}2_3$. These relations yield linear equations in the entries of the submatrices F_i, and we collected enough of these equations to show that they have no solution.

It follows that there is no subgroup $O_8^-(3){}^{\cdot}2_3$, and therefore we have constructed the unique non-split extension $3^{8\cdot}O_8^-(3){}^{\cdot}2_3$ as required.

Having made a faithful 204-dimensional GF(3)-module for $G \cong 3^{8\cdot}O_8^-(3){}^{\cdot}2_3$ we wish also to obtain a 'small' faithful permutation representation. This amounts to finding a subgroup H of low index in G such that $O_3(G) \nleq H$. Such subgroups are not easy to find (see Lemma 1 below).

Now the index 1066 maximal subgroup K of G has structure

$$K \cong 3^{1\cdot 6\cdot 1}.(3^6{:}2^{\cdot}\mathrm{U}_4(3)^{\cdot}(2^2)_{133}) \cong 3^7.(3^6{:}6_2{}^{\cdot}\mathrm{U}_4(3)^{\cdot}(2^2)_{133}),$$

as we see by considering its embedding in both $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$ and $3_+^{1+12\cdot}2^{\cdot}\mathrm{Suz}{:}2$. In order to obtain a subgroup which splits off the central 3-element of $6_2{}^{\cdot}\mathrm{U}_4(3)$, we come down by a further index 252 to the group

$$J \cong 3^7.(3^6{:}(6 \times \mathrm{U}_4(2)){:}2).$$

Now J contains a subgroup $H \cong 3^7.3^6.(2 \times \mathrm{U}_4(2){:}2)$ to index 3 which does not contain $\mathrm{O}_3(G)$. The index of H in G is 805896, and this turns out to be the minimum faithful permutation degree of G.

Lemma 1. The minimal degree of a faithful permutation representation of G is $805896 = 2^3.3^3.7.13.41$. This representation is unique up to permutational isomorphism.

Proof. Since $3^{8\cdot}\mathrm{O}_8^-(3)^{\cdot}2_3$ has a unique minimal normal subgroup the representation we seek is transitive. If there were any smaller or equal sized faithful permutation representation, it would have to come from a subgroup of $\mathrm{O}_8^-(3)^{\cdot}2_3$ of index at most $805896/3 = 268632$. Moreover, this subgroup cannot contain the full Sylow 3-subgroup of $\mathrm{O}_8^-(3)^{\cdot}2_3$, as it is well-known that the Sylow 3-subgroup acts uniserially (apart from two 1s in the middle) on the 3^8 module, and we have already seen that it fails to split off the top factor. The only subgroup of $\bar{K} := K/\mathrm{O}_3(G) \cong 3^6{:}2^{\cdot}\mathrm{U}_4(3){:}2_1$ of index at most $252 = 268632/1066$ that does not contain a Sylow 3-subgroup is $\bar{J} := J/\mathrm{O}_3(G) \cong (3 \times 3^5){:}(2 \times \mathrm{U}_4(2){:}2)$. We need subgroups of index 3 in J that do not contain $\mathrm{O}_3(G)$. But $\bar{J}$ acts on $\mathrm{O}_3(G)$ with a unique maximal submodule, U say; U has dimension 7. So U is a subgroup of any index 3 subgroup of J. In the quotient $J/U \cong (3 \times 3 \times 3^5){:}(2 \times \mathrm{U}_4(2){:}2)$ there are just two normal subgroups of order 3, one of which corresponds to $\mathrm{O}_3(G)$. Thus we obtain that J has just one subgroup of index 3 not containing $\mathrm{O}_3(G)$.

Next consider the index 2214 subgroup $(3^7 \times 3)^{\cdot}\mathrm{O}_7(3){:}2$, which has both quotients $3^{7\cdot}\mathrm{O}_7(3){:}2$ and $3^{\cdot}\mathrm{O}_7(3){:}2$ being non-split extensions. But $\mathrm{O}_7(3)$ has no proper subgroups of index less than 351, see the Atlas [4] for example.

The next maximal subgroup of $\mathrm{O}_8^-(3)^{\cdot}2_3$ is the stabiliser of an isotropic 3-space in the 3^8-module, and has index 22960 and structure $3^{3+6}{:}(\mathrm{L}_3(3) \times \mathrm{Q}_8)$. To get a subgroup of this not containing the full Sylow 3-subgroup, we need index at least 39. But again $22960.39.3 > 805960$.

Similarly, the stabiliser of an isotropic 2-space has index 29848. In order to get a representation on 805896 or fewer points, we need a subgroup of index at most 9 in the corresponding subgroup $3_+^{1+8}{:}(2^{\cdot}\mathrm{S}_4^+ \times \mathrm{M}_{10})$ of $\mathrm{O}_8^-(3)^{\cdot}2_3$. Since subgroups of M_{10} of index at most 9 contain A_6, our subgroup has an A_6 composition factor, and the action of A_6 on 3_+^{1+8} forces our subgroup to contain this too. So our subgroup contains the normal $3_+^{1+8}{:}\mathrm{A}_6$ of $3_+^{1+8}{:}(2^{\cdot}\mathrm{S}_4^+ \times \mathrm{M}_{10})$. The $3_+^{1+8}{:}\mathrm{A}_6$ acts on $\mathrm{O}_3(G) \cong 3^8$ with module structure $(1 \oplus 1) \cdot 4 \cdot (1 \oplus 1)$ (and 12 submodules of dimensions 0, 1, 1, 1, 1, 2, 6, 7, 7, 7, 7, 8). We now claim that $3_+^{1+8}{:}\mathrm{A}_6$ does not split off each of the four 1-dimensional quotients of $\mathrm{O}_3(G) \cong 3^8$ by a maximal submodule. Each of these maximal submodules is fixed by a copy of (an overgroup of) $3^6{:}2^{\cdot}\mathrm{U}_4(3)$ containing the original $3_+^{1+8}{:}\mathrm{A}_6$, and we already know that this lifts to $3^6{:}6_2{}^{\cdot}\mathrm{U}_4(3)$ when we stick this trivial module underneath. Moreover, the subgroup $3^4{:}\mathrm{A}_6$ of $2^{\cdot}\mathrm{U}_4(3)$ is in $3_+^{1+8}{:}\mathrm{A}_6$ and lifts to $3^5{:}\mathrm{A}_6$ in which the A_6 acts uniserially on the 3^5. This proves our claim. Therefore $3_+^{1+8}{:}A_6$ does not split off any non-trivial quotient of $O_3(G) \cong 3^8$, and so no suitable subgroup of G arises from this case.

Finally, in the maximal subgroup $(3^6 \times 3^2){:}L_4(3){:}SD_{16} \cong (3^6 \times 3^2){:}(4 \times L_4(3)).2^2$ the largest candidate subgroup $3^6{:}L_4(3){:}SD_{16}$ has index $9 \times 209223 = 1883007$. □

Remark. The degree 805896 permutation representation of $3^{8\cdot}O_8^-(3)^{\cdot}2_3$ restricts to $3^{8\cdot}O_8^-(3)$ as a sum of two inequivalent [though automorphic] faithful transitive permutation representations of degree 402948. These two representations are the minimal faithful permutations of the Monster type $3^{8\cdot}O_8^-(3)$.

The problem is now to construct this permutation representation explicitly. Our strategy is to look for a suitable object whose stabiliser is our desired point stabiliser H, and then explicitly permute the images of this object. We use the known action on the $805896/3 = 268632$ cosets of J (which is just an action of $O_8^-(3)^{\cdot}2_3$) to help construct a Schreier tree for this permutation representation, in order to make this computation feasible.

First we find generators for the subgroup $3^6{:}(2 \times U_4(2){:}2)$ in $O_8^-(3)^{\cdot}2_3$, ensuring that these generators have $3'$-order. Then we lift to $3^{8\cdot}O_8^-(3)^{\cdot}2_3$ to get generators for J (adjoining extra generators for the 3^8 if necessary). Cubing our generators (and again adjoining elements of the normal 3^7 if necessary) will then give us generators for H. Moreover, we can pick an element $j \in (J \setminus H) \cap 3^8$.

Next we investigate the action of H on the module of shape $8^+ \cdot 195^- \cdot 1^+$ which we have constructed. We look for a submodule fixed by H but not by j. In fact it turned out that it was better to look in the dual module, of shape $1^+ \cdot 195^- \cdot 8^-$ (since 1^+ and 195^- are self-dual while 8^+ and 8^- are dual to each other). By trial and error, we eventually found a submodule of dimension 20 which satisfied these conditions. It is not possible to store all 805896 images of this 20-dimensional submodule in memory at the same time, and a more sophisticated technique is required in order to extract the permutation action of $3^{8\cdot}O_8^-(3)^{\cdot}2_3$ on the images of this submodule.

Now we can make the required permutation action of $O_8^-(3)^{\cdot}2_3$ of degree 268632 easily (for example by working in the degree 1066 permutation representation mentioned earlier), and construct a Schreier tree for it. The first point (the one fixed by the image of J) lifts to three points, which we arbitrarily assign to the 20-dimensional submodule fixed by H (the first of the three points), and its images under j. Attaching three copies of the above Schreier tree to these three points gives us a Schreier tree for the representation on 805896 points.

We then use this Schreier tree to compute the action of $3^{8\cdot}O_8^-(3)^{\cdot}2_3$ in the usual way. For an explanation of how to use Schreier trees in such computations, see Holt, Eick and O'Brien [10, Chapter 4], for example.

11 The group $3_+^{1+12\cdot}2^{\cdot}\mathrm{Suz}{:}2$

The construction of the Monster in [11] began by constructing the subgroup $3_+^{1+12\cdot}2^{\cdot}\mathrm{Suz}{:}2$. However, this construction was not in the form of a conventional matrix or permutation representation. Instead, we constructed two representations (of degrees 180 and 1458 over GF(2)) of a larger group $3_+^{1+12}{:}6^{\cdot}\mathrm{Suz}{:}2$, such that the tensor product of these two representations is the direct sum of two indecomposable modules, one of which supports the action of the given subgroup of the Monster. We can adapt this construction, replacing the 180-dimensional module by the 24-dimensional module for $3^{\cdot}\mathrm{Suz}{:}2$ obtained from the Leech lattice modulo 2. This would give us a representation of degree $24 \times 729 = 17496$ over GF(2). However, this calculation has not been done, as there is a much smaller 3-modular representation, constructed below.

Similarly, it can be shown that the smallest faithful permutation representation of this group is on the $1782 \times 3^{13} = 2841083586$ cosets of the subgroup $2^{\cdot}\mathrm{G}_2(4)^{\cdot}2$. (We omit the proof as it is long and technical.) Clearly this is too large!

To make this group we make modules M_1 and M_2 of shapes $1^- \cdot 64 \cdot 1^+$ and $1^- \cdot 12 \cdot 1^+$ faithfully representing the groups $3^{\cdot}\mathrm{Suz}{:}2$ and $3_+^{1+12}{:}2^{\cdot}\mathrm{Suz}{:}2$. Therefore $M_1 \oplus M_2$ has various subquotients of shape $1^- \cdot (12 \oplus 64) \cdot 1^+$, some of which faithfully represent the Monster subgroup $3_+^{1+12}{}^{\cdot}2^{\cdot}\mathrm{Suz}{:}2$. All modules we consider in this section are constructed over the prime field GF(3). The two faithful 12-dimensional representations 12^+ and 12^- are automorphic, and we make an arbitrary choice in labelling these. We then define module 64^+, representing Suz:2, by $\Lambda^2(12^+) \cong 1^- \cdot 64^+ \cdot 1^-$.

In [11] a 38-dimensional representation of $3_+^{1+12}{:}6^{\cdot}\mathrm{Suz}{:}2$ was made, which in our notation has structure $(1^- \cdot 12^+ \cdot 1^+) \oplus (12^+ \cdot 12^-)$ where $1^- \cdot 12^+ \cdot 1^+$ represents $3_+^{1+12}{:}2^{\cdot}\mathrm{Suz}{:}2$ and $12^+ \cdot 12^-$ represents $6^{\cdot}\mathrm{Suz}{:}2$. Now it turns out that $\Lambda^2(12^+ \cdot 12^-)$ (a $3^{\cdot}\mathrm{Suz}{:}2$-module) has a (unique) submodule of type $1^- \cdot 64^+ \cdot 1^+$. Since we are using standard generators, we know that $[C, DCD]^7$ (with notation as in [11] and [18]) generates the normal subgroup of $3^{\cdot}\mathrm{Suz}{:}2$ of order 3. We check its order in the above representations and find that $1^- \cdot 64^+ \cdot 1^+$ represents $3^{\cdot}\mathrm{Suz}{:}2$ faithfully.

Let $\langle t_1 \rangle$ and $\langle t_2 \rangle$ be the normal cyclic subgroups of order 3 in $6^{\cdot}\mathrm{Suz}{:}2$ and 3_+^{1+12} respectively. In $(1^- \cdot 64^+ \cdot 1^+) \oplus (1^- \cdot 12^+ \cdot 1^+)$ quotient out a diagonal submodule of type 1^- (they are both equivalent under module automorphisms). The group $\langle t_1, t_2 \rangle \cong 3^2$ is still represented faithfully. Now take one of the diagonal submodules of codimension 1. This has the effect of quotienting out by $\langle t_1 t_2 \rangle$ or $\langle t_1 t_2^{-1} \rangle$. One of these is the Monster $3_+^{1+12}{}^{\cdot}2^{\cdot}\mathrm{Suz}{:}2$, and the other one is the $3_+^{1+12}{}^{\cdot}2^{\cdot}\mathrm{Suz}{:}2$ we do not want.

We used the words in [11] to distinguish these cases, noting the correction that J should have been $((\zeta^2\gamma)^4\zeta)^{-1}\zeta(\zeta^2\gamma)^4$.

Remark. An explicit computation with MAGMA [3] shows that the $3^{\cdot}\mathrm{Suz}{:}2$-module $\Lambda^2(12^+ \cdot 12^-)$ has the following structure:

$$\begin{array}{c}
1^- \\
64^+ \\
\hline
\begin{array}{c} 1^+ \oplus 1^- \\ \hline 64^- \\ \hline 1^+ \oplus 1^- \end{array} \quad \oplus \quad 78^- \\
\hline
64^+ \\
1^-
\end{array}$$

12 The group $5^{3+3\cdot}(2 \times \mathrm{L}_3(5))$

Let H be the Monster maximal of shape $5^{3+3\cdot}(2\times \mathrm{L}_3(5))$. The precise structure of this group is hard to pin down, as there are two non-isomorphic groups with such a shape. We shall therefore make both of them, and determine which of the two has Sylow 5-subgroup isomorphic to that of $5_+^{1+6}{:}2^{\cdot}\mathrm{J}_2$.

It is easy to see that in the action of $2^{\cdot}\mathrm{J}_2$ on 5^6 the elements of order 5 centralise 2-spaces, and hence there is no elementary abelian 5^6 in the Monster. But $\mathbb{M}$ has no elements of order 5 in its 31-centraliser, so $\mathrm{O}_5(H) \cong 5^{3+3}$ is special, with centre and derived subgroup of order 5^3.

Let x be an involution in the normal subgroup $5^{3+3}{:}2$ of H. Then a Frattini argument shows that $\mathrm{C}_H(x)$ involves $\mathrm{L}_3(5)$. But Co_1 has no elements of order 31, so $\mathrm{C}_{\mathbb{M}}(x) \cong 2^{\cdot}\mathbb{B}$. Also, $\mathbb{B}$ does not contain $\mathrm{L}_3(5)$ and has Sylow 5-subgroups of order 5^6, so $\mathrm{C}_H(x) \cong 2 \times 5^{3\cdot}\mathrm{L}_3(5)$. Moreover, this group is determined up to isomorphism.

Certainly, $\mathrm{O}_5(\mathrm{C}_H(x)) = \mathrm{Z}(\mathrm{O}_5(H))$, for the case $\mathrm{O}_5(\mathrm{C}_H(x)) \cap \mathrm{Z}(\mathrm{O}_5(H)) = 1$ would force $\mathrm{O}_5(H)$ to be elementary abelian. In particular, $2 \times \mathrm{L}_3(5)$ acts faithfully on the Frattini quotient of $\mathrm{O}_5(H)$, corresponding to the representation $3a^-$ (that is, the tensor product of 'the' natural representation of $\mathrm{L}_3(5)$ by the alternating representation of 2). Moreover, $H/\mathrm{Z}(\mathrm{O}_5(H)) \cong 5^3{:}(2\times \mathrm{L}_3(5))$ is a split extension. Now the exterior square of $3a^-$ is $3b^+$, so this describes the (non-faithful) representation of $2 \times \mathrm{L}_3(5)$ on $\mathrm{Z}(\mathrm{O}_5(H))$.

We need a non-split downward extension of $5^{3a^-}{:}(2 \times \mathrm{L}_3(5))$ by the module $3b^+$. One such extension is the maximal parabolic subgroup $5^{3+3}{:}(2\times\mathrm{L}_3(5))$ of $\mathrm{O}_7(5)$. Another may be obtained as the subdirect product of $5^{3a^-}{:}(2 \times \mathrm{L}_3(5))$ and $5^{3b^+\cdot}\mathrm{L}_3(5)$ over $\mathrm{L}_3(5)$.

On the other hand, an explicit calculation carried out in Magma reveals that $\dim \mathrm{H}^2(5^{3a^-}{:}(2 \times \mathrm{L}_3(5)), 3b^+) = 2$, and so any non-split $5^{3b^+}.(5^{3a^-}{:}(2 \times \mathrm{L}_3(5)))$ can be obtained from these by the following procedure. We create a universal non-split extension $5^{3+3}{:}(2 \times 5^{3\cdot}\mathrm{L}_3(5))$ as the subdirect product of the two given extensions over $5^{3a^-}{:}(2 \times \mathrm{L}_3(5))$. The six non-split extensions are then the quotients of this universal extension by one of the six normal subgroups of order 5^3: two we have already seen, and four 'diagonal' ones. Now $5^{3a^-}{:}(2 \times \mathrm{L}_3(5))$ has an outer automorphism that extends this group to $5^3{:}(4 \times \mathrm{L}_3(5)) \cong \mathrm{AGL}_3(5)$; we may take this automorphism to have order 4 and to commute with $\mathrm{L}_3(5)$. This automorphism extends the two 'basic' versions of $5^{3b^+}.(5^{3a^-}{:}(2 \times \mathrm{L}_3(5)))$ to $5^{3+3}{:}(4 \times \mathrm{L}_3(5)) < \mathrm{SO}_7(5) \cong \mathrm{O}_7(5){:}2$ in which this automorphism negates the 5^{3b^+}, and $5^3{:}(4 \times 5^{3\cdot}\mathrm{L}_3(5))$ in which this automorphism centralises the 5^{3b^+}. It therefore follows that this automorphism does not preserve any of the diagonal groups of shape $5^{3b^+}.(5^{3a^-}{:}(2 \times \mathrm{L}_3(5)))$, and thus swaps them in pairs.

This leads to two possible groups of type $5^{3+3\cdot}(2 \times \mathrm{L}_3(5))$, which we shall show are non-isomorphic, so that only one of them is a subgroup of $\mathbb{M}$.

In order to make explicit representations of these groups, we first make the groups $5^{3b\cdot}\mathrm{L}_3(5)$ and $5^{3+3}{:}(2\times \mathrm{L}_3(5))$ by gluing together representations of $2\times \mathrm{L}_3(5)$ in the by now familiar way. The former can be made by gluing together representations of $\mathrm{L}_3(5)$ of shape $3b \cdot 39b$ and $39b \cdot 1$ to get a representation of shape $3b \cdot 39b \cdot 1$ for a group which does not contain $\mathrm{L}_3(5)$. We think of this as a representation of shape $3b^+ \cdot 39b^+ \cdot 1^+$ for $2 \times 5^{3b^+\cdot}\mathrm{L}_3(5)$.

The other group can be made as follows: first write down the affine group $5^3{:}(2 \times \mathrm{L}_3(5))$ in its natural representation of shape $3a^- \cdot 1^+$. Then choose standard generators for this group, mapping to standard generators of $\mathrm{L}_3(5)$, and satisfying a known presentation. Next we make a representation of shape $3b^+ \cdot 3a^-$ for this group, using the relations in the group to create linear equations for the unknown entries in the matrices. Note however that some of these solutions give rise to representations of $2 \times \mathrm{L}_3(5)$, so we need to ensure that one of the relations in the latter group is *not* satisfied.

Now glue together the two representations of shape $3b^+ \cdot 3a^-$ and $3a^- \cdot 1^+$ for $5^3{:}(2 \times \mathrm{L}_3(5))$, and check that there is no solution to the equations coming from the relations in $5^3{:}(2 \times \mathrm{L}_3(5))$. It follows that the group we have constructed has the shape $5^{3b^+ + 3a^-}{:}(2 \times \mathrm{L}_3(5))$. (To see the splitness of the extension, note that the stabiliser of a vector not in the submodule $3b^+ \cdot 3a^-$ is a complementary $2 \times \mathrm{L}_3(5)$.)

Since our generators in both cases map onto standard generators of $\mathrm{L}_3(5)$, it follows that the direct sum $3b^+ \cdot 39b^+ \cdot 1^+ \oplus 3b^+ \cdot 3a^- \cdot 1^+$ represents our universal extension $5^{3+3}{:}(2 \times 5^{3\cdot}\mathrm{L}_3(5))$, as does any diagonal submodule of codimension 1, of shape $(3b^+ \cdot 39b^+ \oplus 3b^+ \cdot 3a^-) \cdot 1^+$.

We now have four diagonal submodules of type $3b^+$ which we can quotient by, but changing the sign in one factor can be achieved by the known outer automorphism, so it is easy to obtain the two different cases.

At this stage we have two groups, H_1 and H_2, say, of shape $5^{3+3\cdot}(2 \times \mathrm{L}_3(5))$, at least one of which is isomorphic to a maximal subgroup of the Monster. To find out which, shall make suitable permutation representations of the Sylow 5-subgroups of H_1, H_2 and $\mathbb{M}$ and test isomorphism using Magma. This involves converting them into PC-groups and finding their standard presentations.

We already know a suitable permutation representation of the Sylow 5-subgroup of $\mathbb{M}$ on 78125 points, as a subgroup of $5^{1+6}{:}2^{\cdot}\mathrm{J}_2$. Thus the only non-trivial step that is left is to find reasonable permutation representations of H_1 and H_2. In each case we first found an involution mapping to a diagonal involution in $2 \times \mathrm{L}_3(5)$, and found its centraliser (of shape $2 \times 5_+^{1+2}{:}\mathrm{GL}_2(5)$ and index 96875) by the method of [2]. We then used the Magma command CosetImage in the 46-dimensional matrix representation to obtain a permutation representation on 96875 points.

Finally we want some smaller permutation representations (either before or after checking the isomorphism types). This involves finding explicit subgroups to use as point stabilisers and using the Magma command CosetImage again to make the corresponding permutation representation.

We managed to find a core-free subgroup of index 23250 as the normaliser of the fourth power of a certain element of order 20. (In the generators a, b given in [18], the latter element is $(ab)^{20}b$.) Once we obtain the permutation representation of degree 23250, the Magma command Subgroups (with IndexLimit set) will find a core-free subgroup of index 7750.

13 The subgroup $3^{2+5+10\cdot}(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$

The remaining two maximal 3-local subgroups are more complicated, and we could see no easy way to construct them abstractly as matrix or permutation groups. We therefore decided to find them explicitly as subgroups of the Monster constructed in [11], and convert them into permutation groups by permuting a suitable orbit of vectors.

Our strategy is to first find generators for the given subgroup in terms of the generators C, D, E and T of $\mathbb{M}$ defined in [11]. We then find a faithful permutation representation by permuting a suitable orbit of vectors. In general this will not be the smallest faithful permutation representation, which may not be obtainable in this fashion. Therefore we work within the group to find a maximal core-free subgroup H, and use MAGMA to permute the cosets of H to obtain a smaller permutation representation. (We do not claim that our representations are the smallest possible.)

It turns out that the 3-local subgroup of shape $3^2.3^5.3^{10\cdot}(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$ has a (pseudo-primitive) permutation representation of degree $2^4.3^7 = 34992$, on the cosets of a maximal core-free subgroup $3.3^5.3^5.(\mathrm{L}_2(11) \times 2 \times \mathrm{S}_3)$. First we found explicit generators for the full 3-local subgroup: these can be taken as E^{BC}, H, I and T in the notation of [11]. If we want two generators then $E^{BC}H$ and TI will do. We did not use the whole of the subgroup $3^{1+10}{:}(\mathrm{L}_2(11) \times \mathrm{D}_{12})$ to look for suitable vectors, but instead took the smaller subgroup $\langle H, (IHI)^2\rangle \cong \mathrm{L}_2(11)$. This has the advantage that it allows a variety of possible point stabilisers to emerge. There are just four coordinate vectors fixed by this subgroup $\mathrm{L}_2(11)$ in the faithful part of the representation. One of these gave rise to a faithful permutation representation on 69984 points, on the cosets of a subgroup $3_+^{1+10}{:}(\mathrm{L}_2(11) \times \mathrm{S}_3)$. This subgroup, despite appearances, is not contained in our desired point stabiliser $3_+^{1+10}{:}(\mathrm{L}_2(11) \times \mathrm{D}_{12})$. (Indeed, we checked in MAGMA that no action of $3^2.3^5.3^{10\cdot}(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$ on the blocks of a non-trivial block system is faithful. In other words, the action on 69984 points is pseudo-primitive.)

Therefore we make the desired permutation representation in MAGMA by first finding explicit generators for the point stabiliser, and then permuting the cosets of this subgroup. The idea is to take an element of order 3 in the $2^{\cdot}\mathrm{S}_4^+$-factor of a subgroup $\mathrm{L}_2(11) \times 2^{\cdot}\mathrm{S}_4^+$, and take its normaliser. As such an element centralises 3, 3^5 and 3^5 respectively in the three factors of 3^{2+5+10}, we hope that its full normaliser is a group of shape $3_+^{1+10}{:}(\mathrm{L}_2(11) \times \mathrm{D}_{12})$. We take t to be the 22nd power of an element of order 66, thus making sure that t is not in the normal 3^2, and we normalise $\langle t\rangle$. The action on the cosets of this normaliser gives rise to a faithful permutation action of $3^2.3^5.3^{10\cdot}(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$ of degree 34992 on the cosets of $3_+^{1+10}{:}(\mathrm{L}_2(11) \times \mathrm{D}_{12})$. We have also checked in MAGMA that this action is pseudo-primitive.

Remark. Note that the 11-centraliser in our group is $11 \times 3^2{:}2^{\cdot}\mathrm{S}_4^+$. The $3^2{:}2^{\cdot}\mathrm{S}_4^+$ factor centralises only an $\mathrm{L}_2(11)$ in the Monster (see [13]). Therefore there is no complementary $\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+$ in our group. There are, however, complementary M_{11}s, which centralise $3^2{:}\mathrm{SD}_{16}$. The notation 3^{2+5+10} shows the upper and lower central series (which coincide) for the O_3-subgroup, and the subgroup 3^{2+5} is elementary abelian. More details of the structure of this group can be found in [11, 15].

14 The subgroup $3^{3+2+6+6}{:}(\mathrm{L}_3(3) \times \mathrm{SD}_{16})$

For the next case, $3^3.3^2.3^6.3^6{:}(\mathrm{L}_3(3) \times \mathrm{SD}_{16})$, again we first find explicit generators for this subgroup inside the Monster. This subgroup intersects $3^{2+5+10}.(\mathrm{M}_{11} \times 2^{\cdot}\mathrm{S}_4^+)$ in

$$3^{2+5+10}.(3^2{:}\mathrm{SD}_{16} \times 2^{\cdot}\mathrm{S}_4^+) \cong 3^{3+2+6+6}{:}(3^2{:}2^{\cdot}\mathrm{S}_4^+ \times \mathrm{SD}_{16}).$$

Therefore our strategy for making it is to first find this intersection, and then to extend $3^2{:}2{}^{\cdot}\mathrm{S}_4^+$ to $\mathrm{L}_3(3)$. The latter operation can be done inside the centraliser of SD_{16}, so in the centraliser of an element of order 8. Since the latter is a small group, it is easy to find a suitable element extending $3^2{:}2{}^{\cdot}\mathrm{S}_4^+$ to $\mathrm{L}_3(3)$ and normalising our originally chosen 3^3.

Thus we first find the subgroup $3^2{:}\mathrm{SD}_{16}$ of M_{11}, given by the elements $L := H^{IHI}$ and $M := I^{HII}$ found on the M_{11}-page of [18]. (Again we use the notation for the elements $B = D$, C, E, F, G, H, I, K and T as in [11].) Our next job is to find the centraliser in the Monster of the $8H$-element $P := LMG$. We find that $c_3 := CD[P^4, CD]^2$ centralises P^4, and then $c_4 := c_3[P^2, c_3]^2$ and $c_5 := c_3LMc_3(LM)^2[P^2, c_3LMc_3(LM)^2]^2$ centralise P^2. Finally, our subgroup is generated by L, T and $MKE^{BC}(c_5c_4)^2$. (To get two generators, we can replace L and T by LT.)

Remark. The structure of this group is given incorrectly in the Atlas [4], but correctly by Aschbacher [1]. The notation $3^{3+2+6+6}$ reflects the lower/upper central series (which coincide) for the O_3-subgroup. The fact that it is a split extension follows from Norton [13], but we have also verified this computationally.

Next we looked for a suitable subgroup of our group X to be the point stabiliser. Since X must not contain the unique minimal normal subgroup 3^3, we looked for groups X which intersect this 3^3 in 3^2. Therefore X must map into a subgroup $3^2{:}2{}^{\cdot}\mathrm{S}_4^+$ of $\mathrm{L}_3(3)$. The intersection of X with the O_3-subgroup is harder to estimate without detailed knowledge of the structure of this group. However, as in the previous section, we found we did not need any information about this intersection for the next phase of our construction. We simply tried a few collections of elements which might be in a suitable point stabiliser, and looked for fixed coordinate vectors.

We found that the group generated by F, G, $(c_5c_4)^2$ and P^2 fixed four of the fundamental 2-spaces in the 'semi-monomial' action of $3^{12}{:}2{}^{\cdot}\mathrm{Suz}{:}2$ (note that this is in fact a non-faithful action of $3_+^{1+12}{}^{\cdot}2{}^{\cdot}\mathrm{Suz}{:}2$). By taking a coordinate vector in one of these 2-spaces we obtained a permutation action of our group on 227448 vectors. It turns out that this action is pseudo-primitive, so again if we want a smaller permutation representation we have to find our desired point stabiliser explicitly, and permute its cosets. (In fact, we were hoping for a representation on 12636 points, but it seems that no such representation is faithful.)

The smallest faithful representation we could find was on 85293 points. To make the point stabiliser we use the following recipe. First we find an $\mathrm{L}_3(3)$ which lies inside a complementary $\mathrm{L}_3(3) \times \mathrm{SD}_{16}$, by letting t to be the 13th power of an element of order 104, and finding $Y := \mathrm{C}_X(t) \cong \mathrm{L}_3(3) \times 8$.

Take the $3^2{:}2{}^{\cdot}\mathrm{S}_4^+$ in $Y' \cong \mathrm{L}_3(3)$ that stabilises a point in the normal 3^3 of X. Come down to the 3^2 and normalise it. Inside this normaliser we see 3, 3^2, 3^4, 3^2 in the respective factors of $3^{3+2+6+6}$, and $3^2{:}2{}^{\cdot}\mathrm{S}_4^+ \times \mathrm{SD}_{16}$ in the complement, so this normaliser has order $2^8.3^{12}$ and index $3^8.13 = 85293$. Moreover, it does not contain the normal 3^3 of X, so is core-free. Thus we may obtain a faithful, pseudo-primitive, action of X on the cosets of this normaliser.

Remark. A similar process with the other subgroup $3^2{:}2{}^{\cdot}\mathrm{S}_4^+$ in Y' gives an index 2302911 core-free subgroup of X. If u is a $3A$ involution of $Y' \cong \mathrm{L}_3(3)$ then $\mathrm{N}_X(\langle u\rangle)$ is a core-free subgroup of index 113724. The three faithful representations on 85293, 113724 and 227448 points are all available in [18].

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