Examples of 3-dimensional 1-cohomology for absolutely irreducible modules of finite simple groups

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Abstract

It is known that for finite simple groups it is possible for a faithful absolutely irreducible module to have 1-cohomology of dimension at least 3. However, until now no explicit examples have been found. We present two explicit examples where the dimension is exactly 3. It remains an open question as to whether the dimension can be bigger than 3.

1 Introduction

The first cohomology group $\mathrm{H}^1(G;M)$ of a group G at a module M has two fundamental interpretations: first, it parametrises the conjugacy classes of complements of M in the split extension $M{:}G$ of M by G; and second, it parametrises the extensions of M by the trivial G-module. As such it has important applications both in pure group theory and in representation theory. The first cohomology groups of very many modules for finite (especially simple) groups have been calculated over the past thirty years or so (see for example [1, 2, 3, 4, 11, 7, 8]), and they have all turned out to be remarkably small.

We assume that M is faithful and absolutely irreducible, since if either of these conditions fails then $\dim \mathrm{H}^1(G;M)$ can be as large as we like. In most cases which have been calculated until recently, $\mathrm{H}^1(G;M)$ has dimension 0 or 1, and in all the rest it has dimension 2. This led Guralnick and Hoffman [10] to conjecture that for any G and faithful absolutely irreducible M, $\dim \mathrm{H}^1(G;M) \leqslant 2$. However, Scott [13] has shown that this is false, by using deep results in representation theory to show that for q a large enough power of a large enough prime p, the simple group $\mathrm{PSL}_6(q) = \mathrm{L}_6(q)$ has an absolutely irreducible module whose 1-cohomology is at least 3-dimensional. On the other hand, Scott's result does not at present allow one to find an explicit counterexample.

In this note, we describe two counterexamples explicitly. In both cases we prove that $\dim \mathrm{H}^1(G; M) = 3$. As far as we are aware, these are still the only two examples in which it is known that $\dim \mathrm{H}^1(G; M) = 3$. It remains an open question as to whether one can find examples with $\dim \mathrm{H}^1(G; M) > 3$.

The first example we found had $G \cong {}^2\mathrm{E}_6(2)$ and M of dimension 1702 over the field $\mathbb{F}_2$ of two elements. We were attempting to construct a group of shape $2^{2\cdot 2}\mathrm{E}_6(2){:}\mathrm{S}_3$ (isomorphic to a maximal subgroup of the Monster simple group) as a group of 1706×1706 matrices, to put into the Web-Atlas [15], and our first attempt failed. Analysing what went wrong revealed that we had made the apparently reasonable assumption that in this case $\dim \mathrm{H}^1(G; M) = 2$, which on further investigation turned out to be false.

The second example is much smaller, with $G \cong \mathrm{PSU}_4(3) = \mathrm{U}_4(3)$ and M of dimension 19 over the field $\mathbb{F}_3$ of three elements. It turned up during a systematic calculation of $\mathrm{H}^1(G; M)$ for moderate-sized simple groups and all modules M in characteristics p where p^2 divides the order of G. It is now known that $\mathrm{U}_4(3)$ is the smallest group possessing a faithful absolutely irreducible module with 3-dimensional 1-cohomology.

In fact, even 2-dimensional 1-cohomologies do not occur very often. The following is a table of all 2-dimensional 1-cohomologies for absolutely irreducible modules for simple groups of order at most 10^6.

Group	Characteristic	Module(s)
A_6	3	4
A_7	3	13
$\mathrm{L}_3(4)$	2	$9a, 9b$
$\mathrm{L}_3(4)$	3	19
M_{12}	2	10
$\mathrm{U}_3(5)$	5	19
A_9	2	26

These modules are all realisable over the prime field, and the modules are denoted by their dimensions, with a distinguishing letter, if necessary.

Some generic examples of 2-dimensional 1-cohomology are also known. Cline, Jones, Parshall and Scott [11, 7] give the examples of $\Omega_{4m}^{\varepsilon}(q)$ for q even, $q > 2$ and $m \geqslant 2$ acting on a module M of dimension $2m(4m-1) - 2$, where M is the non-trivial composition factor of $\Lambda^2(V)$, with V being a natural module of $\Omega_{4m}^{\varepsilon}(q)$. They also point out that the 'corresponding' (i.e. 26-dimensional) module of ${}^3\mathrm{D}_4(q)$ for q even, $q > 2$ also has 2-dimensional 1-cohomology.

The first author [5] has also noticed a family of cross characteristic examples. Here M is a module of dimension $\frac{q^n-1}{q-1} - 2$ for $\mathrm{L}_n(q)$ over $\mathbb{F}_p$ where $n \geqslant 3$, $p \mid n$ and $q \equiv 1 \pmod{p}$. We obtain M as the non-trivial composition factor of the permutation module of degree $\frac{q^n-1}{q-1}$ of $\mathrm{L}_n(q)$ on the 1-spaces or hyperplanes in the natural module.

2 The first example

In [14] a representation of the sporadic 'Baby Monster' simple group $\mathbb{B}$ was explicitly constructed, in 4370 dimensions over $\mathbb{F}_2$. Using the standard generators a, b defined for $\mathbb{B}$ in [15], it is easy to check computationally that $x := (ab)^{14}ab^2$ has order 38, and that the centraliser in $\mathbb{B}$ of the involution x^{19} is generated by x and $(ax^{19})^3$. This centraliser has shape $2^{\cdot 2}\mathrm{E}_6(2){:}2$.

Now we can use the MeatAxe [12] to analyse the structure of the module restricted to this involution centraliser. We easily find (as described in [14]) a quotient of a submodule, of dimension 1703, which is uniserial with constituents of dimensions 1702 and 1. This module represents ${}^2\mathrm{E}_6(2){:}2$ faithfully, and we may restrict to the simple group ${}^2\mathrm{E}_6(2)$.

In order to adjoin an outer automorphism of order 3 to this group, we first define standard generators c and d by the properties that c is in class $2B$ (as defined in the ATLAS [9]), d is in class $3C$, cd has order 19 and $cdcdcdd$ has order 33. (The elements c and d correspond to the elements a and b on the ${}^2\mathrm{E}_6(2)$ page of the Web-Atlas [15].) Such generators may be easily found using the following procedure. Inside ${}^2\mathrm{E}_6(2){:}2$, any element of order 9 or 18 or 36 powers up to an element in class $3C$. Similarly, any outer element of order 20 powers up to an element in class $2B$. Now conjugate these elements independently 'at random' until the conjugates satisfy the defining properties just listed.

A similar procedure can be used to find another pair of standard generators, which is an image of the first pair under a non-inner automorphism. We find for example that the map $(c,d) \mapsto (c^{dcd}, d^{(cd^2)^6})$ defines an automorphism which has order 3 modulo inner automorphisms. Applying this map to the generating 1703×1703 matrices we obtain three modules of shape $1702 \cdot 1$. We shall show that these three module extensions are linearly independent.

To do this, we apply the standard basis procedure of the MeatAxe to put the 1702-dimensional submodule into standard form. Thus our group generators g $(= c, d)$ are represented in the form $\begin{pmatrix} A(g) & 0 \\ v(g) & 1 \end{pmatrix}$, where $A(g)$ is a 1702×1702 matrix which is the same matrix in all three representations, and only the vector $v(g)$ changes, say $v(g) = v_1(g), v_2(g), v_3(g)$. We may now cut and paste our matrices to create a representation ρ

of degree 1705 as follows:

$$\rho(g) = \begin{pmatrix} A(g) & 0 & 0 & 0 \\ v_1(g) & 1 & 0 & 0 \\ v_2(g) & 0 & 1 & 0 \\ v_3(g) & 0 & 0 & 1 \end{pmatrix}.$$

An easy computer calculation now shows that this module has no 1-dimensional submodule, which proves the claim that $\dim \mathrm{H}^1({}^2\mathrm{E}_6(2); 1702) \geqslant 3$.

Conversely, standard methods using known relations in the group to obtain linear equations for the coordinates of possible vectors $v(g)$ in any module extension 1702.1 quickly show that $\dim \mathrm{H}^1({}^2\mathrm{E}_6(2); 1702) \leqslant 3$. The following relators are sufficient in this case:

$$c^2, d^3, (cd)^{19}, [c,d]^5, (cdcdcd^{-1})^{33}, [c, dcd]^5, ((cd)^6cd^{-1})^8,$$

although they might not suffice to calculate $\dim \mathrm{H}^1({}^2\mathrm{E}_6(2); M)$ accurately for other (irreducible) ${}^2\mathrm{E}_6(2)$-modules M.

3 The second example

This second example was found as a result of systematic computation of cohomologies for small (almost) simple groups. There is a unique absolutely irreducible 19-dimensional module for $\mathrm{U}_4(3)$ in characteristic 3; it is realisable over $\mathbb{F}_3$ and can be obtained as the non-trivial composition factor of $\mathrm{S}^2(V_6)$, where V_6 denotes the natural $\mathbb{F}_3$-module of $\Omega_6^-(3) \cong 2^{\cdot}\mathrm{U}_4(3)$. This time we know a presentation of $\mathrm{U}_4(3)$ on standard generators as defined in the Web-Atlas [15], namely:

$$\langle\, a, b \mid a^2, b^6, (ab)^7, (ab^3)^5, [a,b]^4, [a, b^2ab^2ab^2], (abab^2abab^{-1})^3, (abab^3)^7, (abab^3ab^{-2})^5 \,\rangle.$$

The order of this group can be verified by coset enumeration over $\langle a, b^2\rangle$, since the relators $a^2, b^6, [a, b^2ab^2ab^2]$ force $\langle a, b^2\rangle$ to be an image of $4 \circ \mathrm{SL}_2(3)$. In this case the standard methods give $\dim \mathrm{H}^1(\mathrm{U}_4(3); 19) = 3$. Removing the relators $(abab^3)^7, (abab^3ab^{-2})^5$ from the above presentation of $\mathrm{U}_4(3)$ gives a presentation of $3^{2\cdot}\mathrm{U}_4(3)$. This shorter presentation is useful for calculating 1-cohomology since $\mathrm{H}^1(3^{2\cdot}\mathrm{U}_4(3); M) \cong \mathrm{H}^1(\mathrm{U}_4(3); M)$ for all irreducible $\mathrm{U}_4(3)$-modules M. Indeed $\mathrm{Ext}^1_{3^{2\cdot}\mathrm{U}_4(3)}(M_1, M_2) \cong \mathrm{Ext}^1_{\mathrm{U}_4(3)}(M_1, M_2)$ for all irreducible $\mathrm{U}_4(3)$-modules M_1, M_2 except for some cases in characteristic 3 when $M_1 \cong M_2$. We remark that in this case (unlike the ${}^2\mathrm{E}_6(2)$ case above) the 1-cohomology of the 19-dimensional module can be easily calculated using the CohomologyModule functionality of MAGMA [6].

In fact the module $19 \cdot (1 \oplus 1 \oplus 1)$ for $\mathrm{U}_4(3)$ (with no trivial submodule) occurs 'naturally' as a submodule of the smallest non-trivial permutation module of $\mathrm{U}_4(3)$ over $\mathbb{F}_3$, namely the permutation module of degree 112 on the cosets of $3^4{:}\mathrm{A}_6$. This module has structure (with all factors absolutely irreducible):

$$1 \quad \oplus \quad \left[\begin{array}{c} 19 \\ \hline 1 \\ 1 \oplus 1 \oplus 69 \\ 1 \\ \hline 19 \end{array} \right].$$

The degree 112 permutation representation can be made by taking the action of $\Omega_6^-(3)$ on the isotropic points (i.e. 1-spaces) of V_6. The scalars of $\Omega_6^-(3)$ act trivially in the resulting representation.

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