MTH714U/MTHM024 Group Theory
Exercises 3: November 2009

Hints and solutions

1. *Prove that if the permutation π on n points is the product of k disjoint cycles (including trivial cycles), then π is an even permutation if and only if $n - k$ is an even integer.*

   There are many possible proofs. You can even take this as the definition of the sign of a permutation (see the lecture notes for Algebraic Structures 2).

   We saw that a cycle of even length is an odd permutation, and vice versa, so the sign of a permutation is congruent mod 2 to the sum of the (cycle length -1). In particular, the result is true for the identity permutation. Now suppose we replace k fixed points by a k-cycle. Then the sign of the permutation changes by $k - 1 \bmod 2$. But also the number of cycles changes by $-k + 1$. Hence the result follows by reverse induction on the number of cycles.

2. *Let n be an odd positive integer, $n \geq 3$, and consider the action of the dihedral group D_{2n} of order $2n$ on the n vertices of a regular n-gon.*

   Decide (with proof) for which values of n this action is primitive.

   This action is primitive if and only if n is prime. If n is not prime, say $n = ab$, then you can partition the n vertices into a sets of b vertices, each set forming the vertices of a regular b-gon. This partition is preserved by the action of the whole D_{2n}.

   The converse is a special case of the next question.

3. *Suppose that the finite group G acts transitively on the set Ω, and $|\Omega| = p$, where p is prime.*

   Prove that G acts primitively on Ω.

   The stabilizer of any point is (by the orbit–stabilizer theorem) a subgroup of index p, and is therefore maximal (by Lagrange's Theorem). Therefore the action is primitive.

4. *Let G act transitively on Ω. Show that the average number of fixed points of the elements of G is 1, i.e.*

$$\frac{1}{|G|}\sum_{g\in G}|\{x\in\Omega \mid x^g = x\}| = 1.$$

Count the number of ordered pairs $(x, g) \in \Omega \times G$ such that $x^g = x$ in two ways. On the one hand this is $|G|$ times the average number of fixed points of $g \in G$. On the other hand it is the sum over $x \in \Omega$ of the order of the stabilizer of x, which by the orbit–stabilizer theorem is $|\Omega|$.

5. *(a) Determine the conjugacy classes of the alternating group A_6, and the sizes of these conjugacy classes.*

   (b) Deduce that A_6 is simple.

   (c) Hence write down all the composition series of S_6.

   (a) There is one element with cycle type (1^6), 45 elements with cycle type $(2^2, 1^2)$, 40 with cycle type $(3, 1^3)$, 40 with cycle type (3^2), 90 with cycle type $(4, 2)$, and 144 with cycle type $(5, 1)$. All of these form single conjugacy classes in A_6 except the elements with cycle type $(5, 1)$, which fall into two conjugacy classes of 72 elements each (because each of these elements consists of cycles of distinct odd lengths).

   (b) No non-trivial sum of some of these numbers, including 1, adds up to a divisor of 360. Therefore there is no possibility for a non-trivial normal subgroup.

   (c) The only composition series for S_6 is therefore $1 < A_6 < S_6$.

6. *Let G be the group of rotational symmetries of a regular dodecahedron. Prove that $G \cong A_5$.*

   [Hint: one method is to partition the 20 vertices into 5 sets of 4, each set forming the vertices of a regular tetrahedron.]

   First calculate the order of the group G: any of the 12 faces can be mapped to any other, and the stabilizer of one face consists of the five rotations of that face. Hence $|G| = 12.5 = 60$.

Now there are 20 vertices, and the group acts transitively on them, so the stabilizer of a vertex is of order 3. Geometrically, fixing one vertex, we see three neighbouring vertices, and 6 vertices at 'distance' 2 (in the graph-theoretic sense). Since every vertex has an opposite vertex, this accounts for all 20 vertices.

There are two ways to partition the vertices into five tetrahedra: a 'left-handed' way and a 'right-handed' way. One way is to take a vertex, and walk three steps along the edges, turning left and then right at the two vertices encountered along the way. You will find that the three vertices at the ends of these three paths are connected to each other by the same relation, and therefore the four vertices form a regular tetrahedron, which is determined by any of the four vertices.

[The other way to partition the vertices is to turn first right and then left.]

In either case, we now have an action of G on five objects, and this action is clearly transitive. Fixing one of these objects, you can see its full rotation group A_4 acting on the dodecahedron, and permuting the other four tetrahedra in the natural manner. Therefore there is a homomorphism $G \to A_5$ which is onto and is therefore an isomorphism (since the two groups have the same order).

7. *Let G be the group of permutations of 8 points $\{\infty, 0, 1, 2, 3, 4, 5, 6\}$ generated by $(0, 1, 2, 3, 4, 5, 6)$ and $(1, 2, 4)(3, 6, 5)$ and $(\infty, 0)(1, 6)(2, 3)(4, 5)$. Show that G is 2-transitive. Show that the Sylow 7-subgroups of G have order 7, and that their normalisers have order 21. Show that there are just 8 Sylow 7-subgroups, and deduce that G has order 168. Show that G is simple.*

   Let $\Omega = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$. Action by $(\infty, 0)(1, 6)(2, 3)(4, 5)$ implies that ∞ is in the same orbit as 0, and then action by $(0, 1, 2, 3, 4, 5, 6)$ and its powers implies that Ω is a single orbit, i.e. G is transitive. Now the stabilizer of ∞ contains $(0, 1, 2, 3, 4, 5, 6)$, which, as we have just observed, is transitive on the 7 remaining points. Therefore G is 2-transitive.

   G is a subgroup of S_8, which has order 8.7.6.5.4.3.2, so Sylow subgroups of G have order at most 7. But G has elements of order 7, so its Sylow 7-subgroups have order exactly 7. Inside S_8 the Sylow 7-normalizer has order $7.6 = 42$, e.g. generated by $(0,1,2,3,4,5,6)$, $(1,2,4)(3,6,5)$ and $(1,6)(2,5)(3,4)$. But the last element is an odd permutation, whereas all the generators of G are even permutations. Therefore the normalizer in G of a Sylow 7-subgroup has order 21.

Now we calculate the Sylow 7-subgroups explicitly. Let $S_\infty = \langle(0,1,2,3,4,5,6)\rangle$, and $S_0 = S_\infty^{(\infty,0)(1,6)(2,3)(4,5)} = \langle(\infty,6,3,2,5,4,1)\rangle$ and then inductively $S_t = S_{t-1}^{(0,1,2,3,4,5,6)}$. We show that the set of eight Sylow 7-subgroups $\{S_\infty, S_0, S_1, S_2, S_3, S_4, S_5, S_6\}$ is invariant under conjugation by the generators of G.

Explicit computation shows that conjugation by $(\infty,0)(1,6)(2,3(4,5)$ maps $S_1 \leftrightarrow \langle(0,\infty,5,2,1,4,3)\rangle = S_6$ and similarly $S_2 \leftrightarrow \langle(0,6,4,5,\infty,1,2)\rangle = S_3$ and so on.

Now $|G| = |N_G(S)|$.(number of conjugates of S) = 21.8=168.