MTH714U/MTHM024 Group Theory
Exercises 2: October 2009

Hints and solutions

1. *Use Sylow's theorems to prove that every group of order* 40 *has a proper non-trivial normal subgroup.*

   Deduce that G is soluble.

   The number of Sylow 5-subgroups divides 8 and is congruent to 1 modulo 5, so equals 1. Thus the Sylow 5-subgroup is normal.

   The quotient has order 8, and is therefore nilpotent, hence soluble. Now if N and G/N are soluble, then so is G, by refining the series $1 < N < G$ to a composition series of G, all of whose composition factors are cyclic of prime order.

2. *Do the same for groups of order* 80*.*

   The number of Sylow 5-subgroups is now either 1 or 16. If it is 1, apply the same argument as in the previous question. If it is 16, then we have already accounted for $16.4 = 64$ elements of order 5 in the group, which leaves just 16 other elements. These must form a Sylow 2-subgroup, which is therefore unique, hence normal.

3. *Do the same for groups of order* 84*.*

   The number of Sylow 7-subgroups divides 12 and is congruent to 1 modulo 7, so is 1.

4. *Show that any group of order p^2q, where p and q are distinct primes, has a normal Sylow subgroup.*

   If $q < p$, then there is a unique Sylow p-subgroup.

   If $q > p^2$, then there is a unique Sylow q-subgroup.

   So the interesting case is when $p < q < p^2$, and we may assume the number of Sylow q-subgroups is p^2. Therefore there are $p^2(q-1) = p^2q - p^2$ elements of order q. Every Sylow p-subgroup is therefore a subset of the remaining p^2 elements. Therefore there is only one Sylow p-subgroup and it is normal.

5. *Classify the groups of order pqr, where p, q, r are distinct primes (say $p > q > r$).*

If the number of Sylow p-subgroups is not 1, then it is qr and $p \equiv qr \bmod p$. In this case there are $(p-1)qr = |G| - qr$ elements of order p. In particular, there is not enough room for p or more Sylow q-subgroups, so there is only one Sylow q-subgroup, Q say. Therefore Q is normal, so if P is a Sylow p-subgroup then PQ is a subgroup, of order pq. By discussion in the lectures, this subgroup has a unique Sylow p-subgroup. But this means there can be at most r Sylow p-subgroups in the whole group, which is a contradiction.

Therefore there is a unique Sylow p-subgroup, P say, which is normal. Moreover, if Q is any Sylow q-subgroup and R is any Sylow r-subgroup, then PQ and PR are subgroups, which we know are either cyclic or particular non-abelian groups which we constructed in the lectures.

If they are both cyclic, then there are two possibilities: either $C_p \times C_q \times C_r \cong C_{pqr}$, or in the case when $q \equiv 1 \bmod r$, a group $C_p \times C_q.C_r$.

If PR is cyclic but PQ is not, the only possibility is $PQ \times R$.

If PQ is cyclic but PR is not, there are potentially quite a few possibilities: there is normal subgroup $C_p \times C_q$, and a subgroup C_r which acts non-trivially on the C_p, and may also act in some way on the C_q, if $q \equiv 1 \bmod r$. The number of different groups of this kind is r, because we can choose the action of a generator of C_r on C_p arbitrarily, and then the action on C_q can be any of the r different possibilities. One of these groups is $C_p.C_r \times C_q$, while all the others are of the form $(C_p \times C_q).C_r$.

Finally, if neither PQ nor PR is cyclic, then both Q and R act as automorphisms of P. But the automorphism group of P is cyclic (of order $p-1$), so Q and R commute with each other, which means $QR \cong C_q \times C_r \cong C_{qr}$. The case $p = 7$, $q = 3$, $r = 2$ of this group was discussed in Question 3 of Sheet 1.

You can amuse yourself by writing down all the groups of order 42 (I think there are 6 of them). It gets more complicated when $r > 2$, so you could also try 273.

6. *Write down all the elements of* $\mathrm{Aut}\,(C_2 \times C_2)$*. To which well-known group is it isomorphic?*

   Write $C_2 \times C_2 = \{1, a, b, ab\}$ where $a^2 = b^2 = (ab)^2 = 1$. Then any automorphism must take a to one of the elements of order 2 (that is a, b or ab), and must take b to one of the remaining two. On the other hand, you can see that there is complete symmetry between the three non-trivial elements a, b, ab, so the automorphism group is isomorphic to S_3.

7. *Calculate* $\mathrm{Inn}\,(G)$ *when* $G = D_8$*. Show that* $\mathrm{Aut}\,(G) \cong D_8$*.*

   From the notes, $\mathrm{Inn}\,(G) \cong G/Z(G)$, and $Z(D_8)$ is $C_2 = \{(1,3)(2,4)\}$ (where the vertices of the square are $1, 2, 3, 4$ in cyclic order). Thus you calculate $\mathrm{Inn}\,(G) \cong C_2 \times C_2$.

   Now D_8 is generated by two elements $(1,3)$ and $(1,2)(3,4)$ of order 2 whose product has order 4. The first generator can be mapped to any of the four reflections, and then the second generator must be mapped to one of the two reflections whch does not commute with the first one. This gives an automorphism group of order 8.

   More explicitly, let $\alpha : (1,3) \mapsto (1,2)(3,4) \mapsto (2,4)$ and $\beta : (1,3) \mapsto (1,3), (1,2)(3,4) \mapsto (1,4)(2,3)$. Then you will find that α is an automorphism of order 4, and β is an automorphism of order 2, and that $\alpha\beta = \beta\alpha^{-1}$, so that $\langle \alpha, \beta \rangle \cong D_8$.