Queen Mary
University of London

M. Sci. Examination by course unit 2009

MTH714U Group Theory

Duration: 3 hours

Date and time: 14 May 2009, 1000h–1300h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): R. A. Wilson

Question 1 (a) State the three Sylow theorems, and prove one of them. [12]

(b) Let G be a simple group of order 168.

Show that G has exactly 8 Sylow 7-subgroups, and deduce that G is isomorphic to a subgroup H of the alternating group A_8.

Show that H is 2-transitive but not 3-transitive on $\{1, 2, 3, 4, 5, 6, 7, 8\}$. [13]

Question 2 (a) Define the term *composition series* for finite groups.

Prove that every finite group has a composition series.

State the Jordan–Hölder Theorem for composition series of finite groups. [12]

(b) Let G be the subgroup of A_8 generated by the permutations

$$\begin{aligned} a &= (1,3)(2,6)(4,5)(7,8) \\ b &= (1,2,3,4,5,6,7) \\ c &= (1,2,4)(3,6,5) \end{aligned}$$

Show that the subgroup N of G generated by a and b is normal in G.

Find a non-trivial proper normal subgroup of N, and hence write down a composition series for G.

What is the order of G? [13]

Question 3 (a) State and prove Iwasawa's Lemma. [10]

(b) Define the groups $PSL_n(q)$ [you may assume without proof any properties of finite fields, provided they are stated correctly].

Prove that $PSL_2(q)$ is simple if $q \geq 4$.

Where does your proof break down when $q = 2$ or 3? [15]

Question 4 (a) Define the term *automorphism* of a finite group. Prove that the set $\mathrm{Aut}(G)$ of automorphisms of G is itself a group.

Define the term *inner automorphism*, and prove that the inner automorphisms of G form a group isomorphic to $G/Z(G)$. [8]

(b) Prove that $PSL_2(9) \cong A_6$, but that $PGL_2(9) \not\cong S_6$.

Deduce that S_6 has a non-inner automorphism. [17]

End of Paper