Queen Mary
University of London

M.Sci. EXAMINATION BY COURSE UNITS

MAS428 Group Theory

19 May 2008: 14:30–17:30

The duration of this examination is three *hours.*

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four *questions answered will be counted. Show your calculations.*

Calculators are not *permitted in this examination.*

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

1. (a) [6 marks] State the Sylow theorems.

(b) [6 marks] If $K \lhd G$ and P is a Sylow p-subgroup of K, prove that $G = KN_G(P)$.

(c) [6 marks] Deduce that if p does not divide the index of K in G then G and K have the same number of Sylow p-subgroups.

(d) [7 marks] Calculate the number of Sylow p-subgroups in $GL_2(p)$ and $SL_2(p)$.

2. (a) [2 marks] Define the term *composition series* for a finite group.

(b) [10 marks] State and prove the Jordan–Hölder theorem. [You may assume Zassenhaus's Lemma, provided you state it correctly.]

(c) [4 marks] Prove that if G is a cyclic group of order n and d is a divisor of n then G has exactly one subgroup of order d.

(d) [9 marks] Let G be a cyclic group of order $p_1^{a_1}.p_2^{a_2}.\cdots.p_r^{a_r}$, where $p_1, \ldots, p_r$ are distinct primes and $a_i > 0$. Determine (with proof) the number of distinct composition series for G.

3. (a) [6 marks] Define the *automorphism group* $\mathrm{Aut}(G)$ of a group G, and the *inner automorphism group* $\mathrm{Inn}(G)$, and prove that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

(b) [5 marks] Define the *centre* $Z(G)$ of a group G, and prove that $\mathrm{Inn}(G) \cong G/Z(G)$.

(c) [5 marks] By considering the Sylow 5-subgroups, or otherwise, show that A_5 has a transitive action on 6 points.

(d) [9 marks] Hence, or otherwise, construct an automorphism of S_6 which is not inner.

4. (a) [5 marks] Prove that every finite field has order a power of a prime p.

(b) [5 marks] Explain briefly how to construct a field of order p^d, where p is prime and $d > 1$. (Proofs are not required.) Illustrate with the case $p = 3$ and $d = 2$.

(c) [6 marks] Prove that $PSL_2(4) \cong A_5$.

(d) [9 marks] Prove that $PSL_2(9) \cong A_6$.

[Next question overleaf]

5. (a) [10 marks] Define the groups $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$ and $PSL_n(q)$ and determine their orders.

(b) [5 marks] Explain how to construct a projective plane from a 3-dimensional vector space. If the underlying field has order q, how many points does the projective plane have, and why?

(c) [8 marks] In the action of $PSL_3(2)$ on the projective plane of order 2, show that the action on points is 2-transitive, and prove that the stabilizer of a point is isomorphic to S_4.

(d) [2 marks] Is the same true for the action on lines? Justify your answer.

6. (a) [10 marks] State and prove Iwasawa's Lemma.

(b) [13 marks] Use the action of A_n on unordered triples, and Iwasawa's Lemma, to prove that A_n is simple if $n \geq 7$.

(c) [2 marks] Where does this proof break down if $n = 6$?

[End of examination paper]