Queen Mary
University of London

M.Sci. EXAMINATION BY COURSE UNITS

MAS428 Group Theory

Sample exam (2007): 3 hours

The duration of this examination is three *hours.*

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four *questions answered will be counted. Show your calculations.*

Calculators are not *permitted in this examination.*

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

1. (a) [13 marks] State the three Sylow theorems, and prove one of them.

(b) [12 marks] For each of the three orders 30, 280 and 560 prove that there is no simple group of that order.

2. (a) [5 marks] Define the term *automorphism*, as applied to a group. Prove that the set of automorphisms of a group G itself forms a group.

(b) [5 marks] Define the *inner automorphism group*, and prove that it is a normal subgroup of the automorphism group. Define the *outer automorphism group*.

(c) [5 marks] Let Q_8 denote the group $\{\pm 1, \pm i, \pm j, \pm k\}$ with centre $\{\pm 1\}$ of order 2 and multiplication defined by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Determine its automorphism group. Identify the subgroup of inner automorphisms, and show that its outer automorphism group is isomorphic to S_3.

(d) [10 marks] Prove that the automorphism group of A_5 is isomorphic to S_5.

3. (a) [10 marks] State and prove Iwasawa's Lemma.

(b) [5 marks] Define the projective special linear groups $PSL_n(q)$.

(c) [10 marks] Prove that $PSL_n(q)$ is simple whenever $q > 3$. Where does your proof break down if $q \leq 3$?

4. (a) [10 marks] Define the terms k-*transitive* (where k is a positive integer) and *primitive*, as applied to permutation groups. Prove that every 2-transitive group is primitive, and every primitive group is 1-transitive.

(b) [5 marks] Prove that a transitive permutation group G is primitive if and only if the stabilizer of a point is a maximal subgroup of G.

(c) [10 marks] Let G be the group S_n acting on the set Ω of unordered pairs of integers from $\{1, 2, 3, \ldots, n\}$. Describe the stabilizer of a point in Ω, and calculate its order. Prove that the action of G on Ω is primitive if $n > 4$.

[Next question overleaf]

5. (a) [3 marks] Show that every finite field has order a power of a prime p.

(b) [2 marks] Give an example of a finite field whose order is not a prime.

(c) [10 marks] Let F_q be a finite field of order q. Define the *projective line* over F_q, and explain how $PSL_n(q)$ acts on it. Prove that this action is 2-transitive.

(d) [10 marks] Prove *two* of the following isomorphisms:

(i) $PSL_2(3) \cong A_4$

(ii) $PSL_2(5) \cong A_5$

(iii) $PSL_2(9) \cong A_6$

(iv) $PSL_2(7) \cong PSL_3(2)$

6. (a) [12 marks] Define the term *composition series* of a finite group. State and prove the Jordan–Hölder theorem.

(b) [5 marks] Write down a composition series for the cyclic group of order 63, and deduce a composition series for the dihedral group of order 126.

(c) [8 marks] Compute a composition series for $GL_3(7)$. [You may assume that $PSL_3(7)$ is simple.]

[End of examination paper]