MAS428/MTHM024 Group Theory
Exercises 3: November 2007

1. Prove, by induction on n or otherwise, that the characteristic polynomial of the matrix
$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}$$
is $x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$.

2. Let p be a prime, $p \geq 5$, and let $G = \mathrm{SL}_2(p)$. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generates a Sylow p-subgroup S of G.

   Show that for any $\lambda \in \mathbb{F}_p \setminus \{0\}$ the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ is in $N_G(S)$.

   What is the order of $N_G(S)$, and how many Sylow p-subgroups does G have?

3. Continuing the notation of the previous question, suppose that $p \equiv 1 \pmod{2^k}$, where $k \geq 2$, but $p \not\equiv 1 \pmod{2^{k+1}}$. Show that a Sylow 2-subgroup of $\mathrm{SL}_2(p)$ has order 2^{k+1} and is generated by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where λ is an element of order 2^k in $\mathbb{F}_p$.

   Deduce that a Sylow 2-subgroup of $\mathrm{PSL}_2(p)$ is isomorphic to D_{2^k}.

4. Let G be the group of rotational symmetries of a regular dodecahedron. Prove that $G \cong A_5$.

   [Hint: one method is to partition the 20 vertices into 5 sets of 4, each set forming the vertices of a regular tetrahedron. Another is to label the 6 pairs of opposite faces (suitably) with the points of PL(5), and prove that $G \cong \mathrm{PSL}_2(5)$.]

5. By labelling the 8 Sylow 7-subgroups of $\mathrm{GL}_3(2)$ suitably with the elements $\infty, 0, 1, 2, 3, 4, 5, 6$ of PL(7) construct an isomorphism between $\mathrm{GL}_3(2)$ and $\mathrm{PSL}_2(7)$.

6. Investigate the groups $\mathrm{SL}_3(3)$ and $\mathrm{PSL}_3(3)$ along similar lines to our investigation of $\mathrm{SL}_3(2)$ in the lectures.