MAS428/MTHM024 Group Theory
Exercises 2: October 2007

1. Let G act transitively on Ω. Show that the average number of fixed points of the elements of G is 1, i.e.
$$\frac{1}{|G|}\sum_{g\in G}|\{x\in\Omega \mid x^g = x\}| = 1.$$

2. Prove that if the permutation π on n points is the product of k disjoint cycles (including trivial cycles), then π is an even permutation if and only if $n-k$ is an even integer.

3. Write down all the elements of $\mathrm{Aut}\,(C_2\times C_2)$. To which well-known group is it isomorphic?

4. Calculate $\mathrm{Inn}\,(G)$ when $G = D_8$. Show that $\mathrm{Aut}\,(G)\cong D_8$.

5. Show that $\mathrm{Aut}\,(Q_8)\cong S_4$, where $Q_8 = \langle i, j \mid i^2 = j^2 = (ij)^2\rangle$ is the quaternion group of order 8.

6. Let G be the group of permutations of 8 points $\{\infty, 0, 1, 2, 3, 4, 5, 6\}$ generated by $(0, 1, 2, 3, 4, 5, 6)$ and $(1, 2, 4)(3, 6, 5)$ and $(\infty, 0)(1, 6)(2, 3)(4, 5)$. Show that G is 2-transitive. Show that the Sylow 7-subgroups of G have order 7, and that their normalisers have order 21. Show that there are just 8 Sylow 7-subgroups, and deduce that G has order 168. Show that G is simple.

7. Compute the addition and multiplication tables for the fields
   (a) $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$;
   (b) $\mathbb{F}_8 = \mathbb{F}_2[x]/(x^3 + x + 1)$;
   (c) $\mathbb{F}_9 = \mathbb{F}_3[x]/(x^2 + 1)$.

8. Let $G = \mathrm{GL}_n(q)$. Prove that $Z(G) = \{\lambda I_n \mid 0 \neq \lambda \in \mathbb{F}_q\}$, where I_n is the $n\times n$ identity matrix.

9. Prove that $\mathrm{GL}_2(2)\cong S_3$.

10. Prove that $\mathrm{PGL}_2(3)\cong S_4$ and $\mathrm{PSL}_2(3)\cong A_4$.

11. How many k-dimensional subspaces are there in a vector space of dimension n over the field of q elements?