Large characteristic subgroups satisfying multilinear commutator identities

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If a group G has, say, a nilpotent subgroup H of class c and of finite index n, then G has also a normal nilpotent subgroup of class $\leqslant c$ and of index $\leqslant n!$. But in many situations it is required that this subgroup be normal in a larger group (where G itself is a normal subgroup) — for that we need a characteristic subgroup of G. We can of course consider the automorphic closure $\prod_{\alpha\in \mathrm{Aut}\, G} H^{\alpha}$, which is a characteristic nilpotent subgroup of class $\leqslant nc$. But in an induction on the length of a certain subnormal series it is desirable not to increase the nilpotency class of the subgroup at each step. We prove, in particular, that in the above situation there is a characteristic nilpotent subgroup of the same class $\leqslant c$ whose index is bounded in terms of n and c. Such a result was so far known in folklore only for abelian subgroups, that is, for $c = 1$. Moreover, we prove analogous results for a subgroup satisfying an arbitrary multilinear commutator identity. Multilinear commutator identities define many popular varieties: of nilpotent groups of class c, of soluble groups of derived length d, etc.

Theorem 1: *If a group G has a subgroup H of finite index n satisfying a multilinear commutator identity $\varkappa(H) = 1$, then G has also a characteristic subgroup C satisfying the same identity $\varkappa(C) = 1$ and having finite index bounded in terms of n and the weight of $\varkappa$.*

As an illustration we present a corollary on groups with almost regular automorphisms.

Corollary: *There exist a constant c and a function of a positive integer argument $f(m)$ such that if a finite group G admits an automorphism of order 4 having exactly m fixed points, then G has characteristic subgroups $N \leqslant H \leqslant G$ such that $|G/H| \leqslant f(m)$, the quotient group H/N is nilpotent of class $\leqslant 2$, and the subgroup N is nilpotent of class $\leqslant c$.*

This corollary generalizes Kovács's result on regular automorphism of order 4 and gives an affirmative answer to P. Shumyatsky's question 11.126 in the "Kourovka Notebook"; by the inverse limit argument, it can also be stated about a locally finite group with an element of order 4 with finite centralizer. Earlier we obtained similar results for Lie rings, but for groups we could only obtain a "weak" result, with a bound for the nilpotency class of N depending on m. (Only for a nilpotent 2-group the second author obtained a best-possible result: then the group is almost centre-by-metabelian.) The upgrade from a weak bound for the class of N to a strong one, independent of m, became possible due to Theorem 1.

We also prove similar results for a nilpotent normal subgroup (or an ideal in a Lie algebra) with a bound for the rank of the quotient group (algebra).

Theorem 2: *If a Lie algebra L has a nilpotent ideal of nilpotency class c and of finite codimension r, then L has also an automorphically-invariant nilpotent ideal of class $\leqslant c$ and of finite codimension bounded in terms of r and c.*

Theorem 3: *If a group G has a normal nilpotent subgroup H of class c such that the quotient group G/H has finite rank r and either G is torsion-free or H is periodic, then G has also a characteristic nilpotent subgroup C of class $\leqslant c$ with quotient G/C of finite rank bounded in terms of r and c.*