Notes on permutation characters

pjc, may 2001

This note gives some orbit theorems for subgroups of $\mathrm{P\Gamma L}(n,q)$, proved by using only simple facts about the permutation character. These theorems also follow from Kantor's theorem [1] and Block's Lemma, but the proofs here are more elementary. The theorems also have analogues for subgroups of the symmetric group S_n acting on subsets of $\{1,\ldots,n\}$.

According to the Orbit-counting Lemma, if G acts on Ω with permutation character π, then the number of orbits of G on Ω is equal to $\langle\pi,1\rangle_G$, where 1 denotes the principal character of G. If G also acts on Ω' with permutation character π', then the permutation character of G on $\Omega\times\Omega'$ is $\pi\pi'$, and so the number of orbits of G on $\Omega\times\Omega'$ is $\langle\pi\pi',1\rangle_G=\langle\pi,\pi'\rangle_G$. In particular, the rank of G on Ω is equal to $\langle\pi,\pi\rangle_G$.

Let V be an n-dimensional vector space over $\mathrm{GF}(q)$. For $0\le i\le n$, let P_i denote the set of i-dimensional subspaces of V, and let π_i denote the permutation character of $\mathrm{P\Gamma L}(n,q)$ on P_i.

Lemma 0.1 *There are irreducible characters $\chi_0,\chi_1,\ldots,\chi_{\lfloor n/2\rfloor}$ of $\mathrm{P\Gamma L}(n,q)$ such that*

$$\pi_i=\pi_{n-i}=\chi_0+\chi_i+\cdots+\chi_i$$

for $i\le n/2$.

Proof Let $G=\mathrm{P\Gamma L}(n,q)$. We show first that $\pi_j=\chi_0+\chi_1+\cdots+\chi_j$ for $j\le n/2$. The proof is by induction on j, the result being clear for $j=0$ (since $|P_0|=1$).

We claim that, for $0\le i\le j\le n/2$, we have

$$\langle\pi_i,\pi_j\rangle_G=i+1.$$

Indeed, elementary linear algebra shows that, for $0\le k\le i$, the subset

$$(X,Y)\in P_i\times P_j:\dim(X\cap Y)=k$$

is an orbit; and these are all the orbits. Hence

$$\langle \pi_i - \pi_{i-1}, \pi_j \rangle_G = 1.$$

By the inductive hypothesis, if $i < j$, then $\pi_i - \pi_{i-1} = \chi_i$; so χ_i occurs in π_j with multiplicity 1. We conclude that

$$\pi_j = \chi_0 + \cdots + \chi_{j-1} + \psi$$

for some character ψ containing none of $\chi_0, \ldots, \chi_{j-1}$. The fact that $\langle \pi_j, \pi_j \rangle_G = j+1$ shows that ψ is irreducible. Taking $\chi_j = \psi$, we complete the inductive step.

Now again let $0 \le i \le j \le n/2$. We claim that

$$\langle \pi_i, \pi_{n-j} \rangle_G = i+1 \text{ and } \langle \pi_{n-j}, \pi_{n-j} \rangle_G = j+1.$$

This is proved by linear algebra as before: the orbits on $P_i \times P_{n-j}$ are

$$\{(X,Y) \in P_i \times P_{n-j} : \dim(X \cap Y) = k\}$$

for $k = 0, \ldots, i$, while the orbits on $P_{n-j} \times P_{n-j}$ are

$$\{(X,Y) \in P_{n-j} \times P_{n-j} : \dim(X \cap Y) = k\}$$

for $k = n-2j, \ldots, n-j$.

Then the same argument as before shows that π_{n-j} contains χ_i with multiplicity 1 for $i = 0, \ldots, j$, and nothing else.

We say that a character π of G is *contained in* a character π' if $\pi' = \pi + \psi$ for some character ψ. Now, if π and π' are permutation characters of G on Ω and Ω', and π is contained in π', then:

(a) $\langle \pi, 1 \rangle_G \le \langle \pi', 1 \rangle_G$; that is, the number of orbits of G on Ω' is not less than the number of orbits on Ω;

(b) $\langle \pi, \pi \rangle_G \le \langle \pi', \pi' \rangle_G$; that is, the rank of G on Ω' is not less than the rank on Ω;

Theorem 1 *Let G be any subgroup of $\mathrm{P\Gamma L}(n,q)$, having N_i orbits on P_i for $0 \le i \le n$. Then the following hold:*

(a) $N_i = N_{n-i}$ for $0 \le i \le n/2$.

(b) $N_i \le N_j$ for $0 \le i \le j \le n/2$.

For the Lemma shows that the permutation characters of $\mathrm{P\Gamma L}(n,q)$ on P_i and P_{n-i} are equal, and that the permutation character on P_i is contained in the character on P_j if $0 \le i \le j \le n/2$; these facts remain true when the characters are restricted to the subgroup G. Obviously the analogous statement to (b) also holds if we replace "number of orbits" by "rank".

References

[1] W. M. Kantor, On incidence matrices of projective and affine spaces, *Math. Z.* **124** (1972), 315–318.