Sum-free subsets of a square

pjc

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Let $[n]$ denote the set $\{1,\ldots,n\}$ of natural numbers. A set S of natural numbers is *sum-free* if $x,y \in S$ implies $x+y \notin S$. It is well known that the largest size of a sum-free subset of $[n]$ is $\lceil n/2 \rceil$. Equality is realised by the set of odd numbers in $[n]$, or by the set of numbers strictly greater than $n/2$.

Harut Aydinian asked Oriol Serra: What is the size $f_2(n)$ of the largest sum-free subset of the square $[n]^2 = [n] \times [n]$? (Addition in this case is vector addition.) In particular, is $f_2(n) = c_2 n^2 + O(n)$ for some constant c_2? The purpose of this note is to put upper and lower bounds on the constant c_2. I conjecture that the lower bound is correct, so that $c_2 = 3/5$. (The upper bound $1/\sqrt{\mathrm{e}} = 0.60653\ldots$ is about 1% greater.)

Proposition 1 $f_2(n) \geq \lfloor 3n(n+1)/5 \rfloor$.

Proof Let

$$S = \{(x,y) \in [n]^2 : u \leq x+y \leq 2u-1\}.$$

Clearly S is sum-free. Assuming that $u \leq n+1 \leq 2u-1$, counting by diagonals we find that

$$\begin{aligned} |S| &= (u-1)+u+\cdots+(n-1)+n+(n-1)+\cdots+2(n-u-1) \\ &= \frac{1}{2}\left(-5u^2+(8n+9)u-2(n+1)(n+2)\right). \end{aligned}$$

Clearly $|S|$ is largest when u is the nearest integer to $(8n+9)/10$, that is, $u = \lfloor (4n+7)/5 \rfloor$; and the value of $|S|$ turns out to be $\lfloor 3n(n+1)/5 \rfloor$, as claimed.

Proposition 2 $f_2(n) \leq (1/\sqrt{\mathrm{e}})n^2 + O(n)$.

Proof In this proof I have re-scaled the square $[n]^2$ to the unit square, and neglected boundary effects (which give terms which are $O(n)$). Let (xn, yn) be the element of the sum-free set S whose coordinates have the largest product; let $xy = u$. Now the points above the hyperbola $xy = u$ are excluded, and only half the points in the rectangle with corners at the origin and (x, y) can be included (since we can have at most one of each pair summing to (x, y). This gives an upper bound for the proportion of points to be

$$u + \int_u^1 \frac{u}{x}\,\mathrm{d}x - \frac{u}{2} = \frac{u}{2} - u\log u.$$

This proportion is greatest when its derivative is zero, that is,

$$\frac{1}{2} - \log u - 1 = 0,$$

which occurs when $u = \mathrm{e}^{-1/2}$; the maximum is also $\mathrm{e}^{-1/2}$.

A similar question can be asked for $[n]^d$ for any fixed d, where the number of sum-free sets should be about $c_d n^d$ for some constant c_d. If so, then $c_d \to 1$ as $d \to \infty$, as the following shows.

Proposition 3 *There is a sum-free set of size at least* $\left(1 - \sqrt{3/d}\right) n^d$ *in* $[n]^d$.

Proof The set

$$\left\{ x \in [n]^d : \frac{d(n+1)}{3} \le \sum x_i < \frac{2d(n+1)}{3} \right\}$$

is obviously sum-free. Rather than count this set exactly, we observe that its size is cn^d, where c is the probability that $dn/3 \le \sum x_i < 2dn/3$, where $x_1, \dots, x_d$ are independent random variables uniformly distributed on $[n]$. We have $E(x_i) = (n+1)/2$ and $\mathrm{Var}(x_i) = (n^2-1)/12$. So $E(\sum x_i) = d(n+1)/2$ and $\mathrm{Var}(\sum x_i) = d(n^2-1)/12$. By Chebychev's inequality,

$$P\left(\left|\sum x_i - d(n+1)/2\right| \ge \frac{d(n+1)}{6} \right) \le \frac{\sqrt{d(n^2-1)/12}}{d(n+1)/6} < \sqrt{\frac{3}{d}}.$$

Finally, the same techniques that were used for the square can be applied to the cube with a little more work. The result will be stated here without proof.

Proposition 4 $(10 + \sqrt{15})/20 = 0.69365\ldots \le c_3 \le 2/\mathrm{e} = 0.73576\ldots.$