Problems from the DocCourse: Day 7

Cycle index and orbit counting

Recall that, if G is an oligomorphic permutation group, then $\tilde{Z}(G)$ is the sum of the cycle indices of $G(A)$, where A runs over a set of orbit representatives on finite sets and $G(A)$ is the group of permutations of A induced by its setwise stabiliser. Also $F_G(x)$ is the exponential generating function for the number of orbits of G on n-tuples of distinct points, and $f_G(x)$ the ordinary generating function for the number of orbits of G on n-sets. They are obtained from $\tilde{Z}(G)$ by the specialisations $s_1 \leftarrow x$, $s_i \leftarrow 0$ for $i > 1$, resp. $s_i \leftarrow x^i$ for all i.

$F_G^*(x)$ is the exponential generating function for the number of orbits of G on all n-tuples (repeats allowed).

1. Let B be the group of permutations which preserve or reverse the order of the rational numbers. Calculate $\tilde{Z}(B)$, and hence evaluate $F_B(x)$ and $f_B(x)$. Compare these with what you expect.

Is is possible to assign a value fo $F_B(-1)$?

2. Same question for D, the group of permutations which preserve or reverse the circular order on the set of complex roots of unity.

3. Let G be the permutation group induced by the symmetric group S_n acting on the set of all 2-element subsets of $\{1,\ldots,n\}$.

- Show that the specialisation $s_i \leftarrow 1+x^i$ gives a polynomial p in which the coefficient of x^m is the number of graphs on n vertices and m edges, up to isomorphism.
- Let g be an element of S_n having c_i cycles of length i (in the usual action). Show that each cycle of length i contributes $(i-1)/2$ cycles of length i (if i is odd) or $(i-2)/2$ cycles of length i and one of length $i/2$ (if i is even) in the action on 2-sets. Show also that each pair consisting of cycles of lengths i and j contributes $\gcd(i,j)$ cycles of length $\operatorname{lcm}(i,j)$.
- Hence calculate the cycle index of S_4 on 2-sets, and enumerate the 4-vertex graphs by number of edges. Check by listing the graphs.

4. Show that the limit of the proportion of derangements in the group S_n acting on 2-sets, as $n \to \infty$, is $2e^{-3/2}$.

Is it possible to assign a value to $F_G(-1)$, where G is the infinite symmetric group in its action on 2-sets? [There is no answer to this question!]

Calculate the limiting proportion of derangements of S_n acting on 3-sets as $n \to \infty$.

5. Let $G = C_2 \operatorname{Wr} A$, where C_2 is the cyclic group of order 2 acting regularly, and A is the group of order-preserving permutations of $\mathbb{Q}$. Show that $f_n(G)$ is the nth Fibonacci number.

6. Let $F_n^*(G)$ denote the number of orbits of G on all n-tuples of points of Ω (repetitions allowed). Prove that

$$F_n^*(G) = F_n(G \operatorname{Wr} S).$$

7. For $i = 1, 2$, let G_i be an oligomorphic permutation group on Ω_i. Let $G = G_1 \times G_2$ acting *coordinatewise* on $\Omega_1 \times \Omega_2$, that is, $(\omega_1, \omega_2)(g_1, g_2) = (\omega_1 g_1, \omega_2 g_2)$. Prove that

$$F_n^*(G_1 \times G_2) = F_n^*(G_1) F_n^*(G_2).$$

Prove that, with this action, $F_n(A \times A)/n!$ is equal to the number of matrices with entries 0 and 1 which have exactly n ones and have no row or column consisting entirely of zeros.

Hence find a formula for this number.