Problems from the DocCourse: Day 6

Properties of the random graph

In the following, R is the unique countable "random graph".

1. Prove that a countable graph G is embeddable as a *spanning subgraph* of R (using all of the vertices and some of the edges) if and only if G has the property that, given a finite set V of vertices, there is a vertex z joined to none of the vertices in V.

2. Prove that, given R, each of the following operations produces a graph which is isomorphic to R:

- deletion of a finite number of vertices;
- addition or removal of a finite number of edges;
- *switching* with respect to a finite set A of vertices (that is, interchange adjacency and non-adjacency between A and its complement, leaving edges within or outside A unchanged).

3. Let $\mathrm{AAut}(R)$ be the group of "almost automorphisms" of R, that is, permutations which map edges to edges and non-edges to non-edges with finitely many exceptions. Show that $\mathrm{AAut}(R)$ is highly transitive and contains no finitary permutations.

4. Write the natural numbers in base 2, and concatenate them into a single binary string

$$s = (011011100101\ldots).$$

Form a graph with vertex set $\mathbb{Z}$, in which x and y are joined if and only if $s_i = 1$, where $i = |y-x|$. Show that this graph is isomorphic to R.

A highly transitive free group

This construction is due to Kantor, based on an idea of Tits.

Let F be a free group with countably many generators $f_1, f_2, \ldots$. [This means that these elements generate F, and no non-trivial word in the generators and their inverses is equal to the identity.]

Embed F in $\mathrm{Sym}(\Omega)$ by its regular representation, where $\Omega = F$. Let $N = \mathrm{FSym}(\Omega)$.

Enumerate all pairs (s,t), where s and t are tuples of distinct elements of Ω of the same length: $(s_1,t_1),(s_2,t_2),\ldots$. Since N is highly transitive, choose an element $n_i \in N$ mapping $s_i f_i$ (the image of s_i under f_i) to t_i, for each i. Let G be the group generated by $f_1 n_1, f_2 n_2, \ldots$. . Prove that

- G is highly transitive on Ω;
- every non-identity element of G *fixes* only finitely many points of Ω (that is, G is *cofinitary*);
- G is a free group with the given elements as generators.