Examination: Permutation groups, polynomials and structures

You should try to do at least three questions.

1. Let G be a transitive permutation group on the set Ω, and suppose that G has a regular normal subgroup N.

Show that we can identify Ω with N such that

- N acts by right multiplication;
- G_α acts by conjugation (if α is the point of Ω identified with the identity element of N).

Hence show that, if G is 2-transitive and N is finite, that all non-identity elements of N have the same order, and hence that N is an elementary abelian p-group (a product of copies of the cyclic group of order p) for some prime p.

2. What is an *oligomorphic* permutation group?

Let A be the group of all order-preserving permutations of the ordered set $\mathbb{Q}$. Prove that A is transitive on the set of n-subsets of $\mathbb{Q}$ for all $n \in \mathbb{N}$.

Let G be the wreath product $S \operatorname{Wr} A$, where S is the infinite symmetric group. Show that the orbits of G on n-sets correspond bijectively to expressions

$$n = a_1 + a_2 + \cdots$$

where $a_1, a_2, \ldots$ are positive integers. Show that the number of such expressions is 2^{n-1}, and deduce that G has 2^{n-1} orbits on n-sets.

3. State and prove the *Orbit-Counting Lemma*.

Let A be a finite set of size q, and A^n the set of all n-tuples of elements of A. Let the symmetric group S_n act on A^n by permuting the coordinates. Show directly that the number of orbits of S_n on A^n is equal to $\binom{n+q-1}{n}$. Show that a permutation in S_n which has k cycles fixes q^k elements of A^n. Now show, using the Orbit-Counting Lemma, that, if $a(n,k)$ is the number of permutations in S_n which have k cycles, then

$$q(q+1)\cdots(q+n-1) = \sum_{k=1}^{n} a(n,k) q^k.$$

4. State *Fraïssé's Theorem* on the existence of countable homogeneous relational structures.

Let $\mathcal{T}$ be the class of all finite tournaments. (A tournament consists of a set of vertices with a directed arc in just one direction between each pair of distinct vertices.) Show that $\mathcal{T}$ is a Fraïssé class. Let T be its Fraïssé limit, and G its automorphism group. Show that G is a 2-set-transitive permutation group which is not 2-transitive, and that G contains no permutations of order 2. Show also that the number of orbits of G on ordered n-tuples of distinct vertices of T is $2^{n(n-1)/2}$.

5. Let G be the group of rotations and reflections of a 3×3 square grid; G is the permutation group on the nine squares of the grid in the picture, generated by $(1,3,9,7)(2,6,8,4)$ and $(1,3)(4,6)(7,9)$.

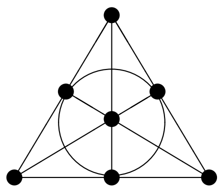
7	8	9
4	5	6
1	2	3

Calculate the cycle index of G.

The nine squares are replaced by panes of glass, some red and some blue. Calculate the generating function for the number of patterns which can be obtained with k red squares, up to rotation and reflection.

How many patterns are there if two panes sharing an edge must have different colours?

6. The following picture shows the *projective plane of order* 2.



This represents a matroid of rank 3, whose bases are all the sets of three points not forming a line of the projective plane.

Calculate the Tutte polynomial of the matroid.

There is a binary code associated with the matroid. Calculate its weight enumerator.

Is it true that this matroid arises from an IBIS permutation group?