Synchronizing non-uniform maps

Peter J. Cameron, CSG, January 2013

Joint work with João Araújo

1 Synchronization

Let Ω be a set with n elements. A *transformation monoid* on Ω is a set of maps from Ω to itself which is closed under composition and contains the identity. A transformation monoid M is *synchronizing* if it contains a map with rank 1 (that is, whose image has only one element).

The notion comes from automata theory. A (finite deterministic) automaton consists of a set Ω of *states* and a set S of *transitions* or maps on Ω. It is said to be synchronizing if there is a word in the transitions which evaluates to a map of rank 1. (Such a word is called a *reset word*.) So (Ω, S) is synchronizing if and only if the monoid $M = \langle S \rangle$ generated by S is synchronizing.

It is easy to tell whether an automaton is synchronizing, but hard to say what the length of the shortest reset word is.

2 Monoids and graphs

There is a very close connection between transformation monoids and graphs.

In one direction, let Γ be a graph. An *endomorphism* of Γ is a map on the vertex set of Γ which maps edges to edges. The set $\mathrm{End}(\Gamma)$ of endomorphisms of Γ is a transformation monoid.

In the other direction, let M be a transformation monoid on Ω. Form a graph $\Gamma = \mathrm{Gr}(M)$ on the vertex set of Ω by the rule that v and w are joined if and only if there is no element $f \in M$ such that $vf = wf$.

We have $M \leq \mathrm{End}(\mathrm{Gr}(M))$ for any transformation monoid M. For suppose that $f \in M$ and $\{v, w\}$ is an edge of $\mathrm{Gr}(M)$. By definition, $vf \neq wf$. If

$\{vf, wf\}$ were a non-edge, then there would exist $h \in M$ with $(vf)h = (wf)h$; but then $fh \in M$ and $v(fh) = w(fh)$, contradicting the assumption that $\{v, w\}$ is an edge. So $\{vf, wf\}$ is an edge.

$\Gamma = \mathrm{Gr}(M)$ has the property that its clique number $\omega(\Gamma)$ and chromatic number $\chi(\Gamma)$ are equal. For let f be an element of minimal rank in M. Then the image of f cannot be further compressed, and so is a clique in $\mathrm{Gr}(M)$; and the kernel of f (the partition of Ω into inverse images of points in the image) is a proper colouring, since no part can contain an edge. Clearly the cardinalities of image and kernel are equal.

We see that there is only one obstruction to synchronization:

Theorem 1 *A transformation monoid M is non-synchronizing if and only if there is a non-null graph Γ on M, with $\omega(\Gamma) = \chi(\Gamma)$, such that $M \le \mathrm{End}(\Gamma)$.*

Proof If $M \le \mathrm{End}(\Gamma)$ for some non-null graph Γ, then M is not synchronizing: an edge cannot be collapsed. For the converse, take $\Gamma = \mathrm{Gr}(M)$.

3 Permutation groups

A permutation group G on Ω is a subgroup of the symmetric group, that is, a transformation monoid whose elements are permutations.

A permutation group G is *transitive* if, for any $v, w \in \Omega$, there exists $g \in G$ with $vg = w$; it is *primitive* if it is transitive and preserves no non-trivial equivalence relation on Ω; and it is 2*-transitive* if it acts transitively on the set of ordered pairs of elements of Ω.

If G is transitive but imprimitive, a part of a non-trivial G-invariant partition is called a *block of imprimitivity* of G.

Let f be a map on Ω which is not a permutation. We say that G *synchronizes* f if $\langle G, f\rangle$ is synchronizing. By abuse of terminology, we say that the group G is *synchronizing* if it synchronizes every non-permutation.

Now G is non-synchronizing if and only if it is a group of automorphisms of a graph Γ, not complete or null, with $\omega(\Gamma) = \chi(\Gamma)$. Hence:

- A 2-transitive group is synchronizing (since it is not a group of automorphisms of any graph except the complete and null graphs).
- A synchronizing group is primitive (since an imprimitive group preserves the graph whose connected components are complete graphs on the parts of the invariant partition, and so is not synchronizing).

Neither of these implications reverses. However, Araújo has made the following bold conjecture. A map is *uniform* if all parts of its kernel have the same size, and is *non-uniform* otherwise.

Conjecture A primitive group synchronizes any non-uniform map.

The rest of this note considers some cases of this conjecture.

We define the *kernel type* of a map f to be the partition of n given by the sizes of the parts of the kernel of f.

4 Maps of large rank

The following theorem was proved by Rystsov:

Theorem 2 *A permutation group G of degree n is primitive if and only if it synchronizes every map of rank $n-1$.*

Our first result is a strengthening of this theorem.

Theorem 3 *Let G be a permutation group of degree n, and $1 < k < n$. Then G is imprimitive with a block of imprimitivity of size at least k if and only if it fails to synchronize some map whose kernel has a part of size k and all other parts singletons.*

We note that a map of rank $n-1$ satisfies the conditions of this theorem, with $k = 2$.

Proof Suppose first that G is imprimitive and has an invariant equivalence relation with parts of size at least k. Let Γ be the complete multipartite graph in which all pairs of vertices in different parts are edges. Then any map which maps each part into itself is an endomorphism; we can choose an endomorphism which collapses k points in a single part to one of them, and maps every further point to itself.

Conversely, suppose that G is a transitive permutation group, f is a map whose kernel has type $(k, 1, \ldots, 1)$, and suppose that G fails to synchronize f. Let $\Gamma = \mathrm{Gr}(\langle G, f\rangle)$, and let A be the part of size k of the kernel of f. There are no edges within A; so, if $v \in A$, then the set $N(v)$ of neighbours of v is mapped bijectively by f to the set $N(vf)$ of neighbours of vf. The same is true for another point $w \in A$; so $N(v) = N(w)$.

Now define an equivalence relation $\equiv$ on Ω by the rule that $v \equiv w$ if and only if $N(v) = N(w)$. This relation is G-invariant, since $G \leq \mathrm{Aut}(\Gamma)$, and is non-trivial (with equivalence classes of size at least k) since A is contained in a single equivalence class. So the theorem is proved.

With a little more effort we can directly improve Rystsov's Theorem:

Theorem 4 *A primitive group of degree n synchronizes any map of rank $n-2$.*

Proof If f has rank $n-2$, then its kernel type is either $(3,1,\ldots,1)$, or $(2,2,1,\ldots,1)$. The first case is dealt with by the preceding theorem. In the second case, more careful analysis shows that the permutation which interchanges the two points in each kernel class of size 2 and fixes all other points is an automorphism of the graph $\Gamma = \mathrm{Gr}(\langle G, f\rangle)$. Now a primitive group of degree greater than 8 which contains a double transposition is known to be symmetric or alternating, and hence 2-transitive (and hence synchronizing). All exceptions of smaller degree are known, and the theorem is easily checked for these.

5 Maps of small rank

At the other end of the scale, we have the following result, first observed by Neumann:

Theorem 5 *A primitive group of degree $n > 2$ synchronizes every map of rank 2.*

Proof A non-null graph which has an endomorphism onto a subgraph of size 2 is bipartite. If it is disconnected, its connected components form blocks of imprimitivity; if it is connected, then its bipartite blocks are blocks of imprimitivity.

Extending this to higher rank looks difficult, for two reasons. First, there will be counterexamples. The group $S_3 \mathrm{wr} S_m$ of degree 3^m (the automorphism group of the m-dimensional hypercube with three points on each edge) is primitive but fails to synchronize a suitable map of size 3. Second, graphs with chromatic number 3 are much more difficult than graphs with chromatic number 2. Nevertheless, we were able to make some progress.

First some general observations. Let M be a transformation monoid which contains a transitive permutation group G, and let f be an element of M with minimum rank r. As we have seen, the graph $\mathrm{Gr}(M)$ has clique number and chromatic number r: the image of f is an r-clique, and f itself is an r-colouring. Now f must be uniform. For each part of the kernel of f is an independent set in $\mathrm{Gr}(M)$, which is vertex-transitive; and in a vertex-transitive graph, the product of clique number and independence number is at most the number of vertices.

Theorem 6 *Let M be a transformation monoid containing a primitive permutation group. Let r be the minimum rank of an element of M, and let f be an element of rank r, where $r > 1$. Then M cannot contain an element with rank greater than r, all but one of whose kernel classes are kernel classes of f.*

The proof depends on the following result. A *primitive graph* is a graph whose automorphism group is primitive.

Theorem 7 *Let Γ be a primitive graph with chromatic number r. Then Γ cannot contain a subgraph consisting of a complete graph of size $r+1$ with an edge deleted.*

Proof Suppose that we have such a subgraph on the vertex set $\{1, 2, \ldots, r+1\}$, where all edges except $\{r, r+1\}$ are present. Then $\{1, \ldots, r\}$ is a complete graph, so all its vertices have different colours (in a fixed colouring c of Γ with r colours); and the same is true for $\{1, 2, \ldots, r-1, r+1\}$. So $c(r) = c(r+1)$. The same argument applies for the image of this subgraph under any element of the primitive group G.

Now let Δ be the graph whose edge-set is the G-orbit of $\{r, r+1\}$. Then Δ is non-null and G-invariant, and any edge of Δ is contained within a colour class of c. But this contradicts primitivity, since the connected components of Δ are blocks of imprimitivity for G.

Proof of Theorem 6 Suppose that the hypotheses of this theorem hold, and let h be an element of m with rank $r + k - 1$ (for $k > 1$) whose kernel has k parts $A_{1,1}, \dots, A_{1,k}$ which partition a part A_1 of the kernel of f, and the remaining parts of the kernels of f and h agree. Let $a_{1,i} = A_{1,i}h$, and $a_j = A_j h$ for $j > 1$.

Note that, if B is a part of the kernel of f, and $v \notin B$, then there is an edge from v to a point of B. For if not, then $B \cup \{v\}$ is an independent set of size $n/r + 1$, contradicting the bound for clique number and independence number in a vertex-transitive graph. Hence there are edges between any two of A_j and A_k for $j, k > 1$, and from any $A_{1,i}$ to any A_j with $j > 1$. Since h is a graph endomorphism, we see that the induced subgraph on $\{a_{1,1}, a_{1,2}, a_2, \dots, a_r\}$ is a complete graph with an edge removed, contrary to the preceding theorem.

From this we can deduce:

Theorem 8 *Let M be a transformation monoid containing a primitive permutation group, and let r be the minimum rank of an element of M, with $r > 1$. Then M cannot contain an element of rank $r + 1$.*

For we may assume that the kernel of such an element h refines the kernel of an element f of minimum rank r (by replacing f by hf if necessary); and then only one kernel class of f is split by the kernel of h, contradicting the theorem.

Finally, we have our result about maps of small rank:

Theorem 9 *A primitive group synchronizes any non-uniform map of rank* 3 *or* 4.

Proof Suppose that G is primitive and does not synchronize h, of rank 3 or 4. Now an element of minimum rank in $\langle G, h\rangle$ is uniform, so has rank 2 or 3; if rank 2, then G is imprimitive by Neumann's result, while rank 3 contradicts the preceding theorem.