Matrix stability from bipartite graphs (Murad Banaji, University of Portsmouth)

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1. We can regard a matrix as:

   - a linear algebraic object, i.e. a representation in particular coordinates of some linear map between vector spaces. We then focus on properties invariant under coordinate transformation. E.g. for $\mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$ **isomorphism = similarity** and we are interested in the characteristic polynomial, spectrum and other such invariants. Given $A, B \in \mathbb{R}^{n \times n}$, we write $A \overset{\text{sim}}{\sim} B$ for $A = M^{-1}BM$. Given $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ we write

   $$\text{Sim}(\mathcal{A}) = \{B \colon B \overset{\text{sim}}{\sim} A \in \mathcal{A}\}, \quad \text{Spec}\, \mathcal{A} = \bigcup_{A \in \mathcal{A}} \text{Spec}\, A\,.$$

   - a combinatorial object; we define various measures of structure, perhaps via other discrete objects. The natural notion of **isomorphism is no longer similarity**, but depends on our characterisation of structure.

2. What might we mean by **structure**? For example, for $A \in \mathbb{R}^{n \times m}$, we might identify "structure" with "sign-pattern": we define a $(-1, 0, 1)$ matrix $\text{sign}\, A$ and say that $A \overset{\text{sign}}{\sim} B$ if $\text{sign}\, A = \text{sign}\, B$. Here the equivalence class $[A]$ is its **qualitative class**:

   $$\mathcal{Q}(A) = \{B \in \mathbb{R}^{n \times m} \colon B \overset{\text{sign}}{\sim} A\}$$

   ($\overline{\mathcal{Q}(A)}$, its closure, is also important.) For square matrices we might define isomorphism via: $A \cong B$ if $\text{sign}\, A = T^t(\text{sign}\, B)T$ for some permutation matrix T.

3. Given $A \in \mathbb{R}^{n \times n}$, define G_A, **the signed digraph of** A, via its adjacency matrix:

   $$A(G_A) = \text{sign}\, A\,.$$

   Clearly $G_A \cong G_B$ (in the natural sense) iff $A \cong B$ (in the above sense).

4. So, in general, combinatorial isormorphism is **neither necessary nor sufficient** for linear algebraic isomorphism: $A \overset{\text{sign}}{\sim} B \not\Rightarrow A \overset{\text{sim}}{\sim} B$ and $A \overset{\text{sim}}{\sim} B \not\Rightarrow A \cong B$. I.e., there are non-similar matrices with identical sign-patterns and similar matrices with non-permutation-similar sign-patterns.

5. The goal here: use structure to make claims about **spectrum**. Why? Original motivation from dynamical systems and bifurcation theory; we may be able to make claims about the possibility of various local bifurcations, or even global claims about the injectivity of functions. Structural claims are "robust" (satisfied by families of models); and precise as the rely on finite exact computations.

An example and some basic notions

6. A circuit (directed cycle) in G_A is odd if it contains an odd number of positive arcs (we'll need to generalise later, and see then that "odd" = "1-odd" in the more general sense). $A \in \mathbb{R}^{n \times n}$ is **sign nonsingular** ("$A \in \textbf{SNS}$") if all matrices in $\mathcal{Q}(A)$ are nonsingular.

7. For $A \in \mathbb{R}^{n \times n}$ with $A_{ii} > 0$, $A \in \textbf{SNS}$ if and only if all circuits in G_A are odd ("$G_A \in \textbf{ODD}$"). This is a "structure to spectrum" result: if $A \in \mathbb{R}^{n \times n}$ with $A_{ii} > 0$ then

   $$\boxed{G_A \in \textbf{ODD} \Leftrightarrow \text{Spec}\, \mathcal{Q}(A) \subseteq \mathbb{C} \backslash \{0\}}$$

8. Actually we can do better. $A \in \mathbb{R}^{n \times n}$ is a P-matrix ("$A \in \mathbf{P}$") if all its principal minors are positive. A is a P_0-matrix ("$A \in \mathbf{P_0}$") if all its principal minors are nonnegative.

9. This is a structural characterisation ($\mathrm{Sim}(\mathbf{P}) \not\subseteq \mathbf{P}$), with spectral implications: by a result of Kellogg [1], the spectra of $n \times n$ P-matrices avoid the following closed "wedge" in $\mathbb{C}$.

$$F(n) = \{re^{i\theta} \in \mathbb{C} \colon |\theta - \pi| \leq \pi/n\}.$$

10. Easy: $(A \in \mathbf{SNS}, A_{ii} > 0) \Leftrightarrow \mathcal{Q}(A) \subseteq \mathbf{P}$. So we really have for $A \in \mathbb{R}^{n \times n}$ with $A_{ii} > 0$:

$$\boxed{G_A \in \mathbf{ODD} \Leftrightarrow \mathcal{Q}(A) \subseteq \mathbf{P} \Leftrightarrow \mathrm{Spec}(\mathcal{Q}(A)) \subseteq \mathbb{C}\backslash F(n).}$$

More generally for $A \in \mathbb{R}^{n \times n}$ (with diagonal elements not necessarily positive):

$$\boxed{G_A \in \mathbf{ODD} \Leftrightarrow \mathcal{Q}(A) \subseteq \mathbf{P_0} \Leftrightarrow \mathrm{Spec}(\mathcal{Q}(A)) \subseteq \overline{\mathbb{C}\backslash F(n)}.}$$

11. **Remark:** the implication that (assuming positive diagonal entries) $\mathrm{Spec}(\mathcal{Q}(A)) \subseteq \mathbb{C}\backslash F(n) \Rightarrow \mathcal{Q}(A) \subseteq \mathbf{P}$ takes a little proof: in fact it can be shown that $(A \in \mathbb{R}^{n \times n}, A_{ii} > 0, \mathcal{Q}(A) \not\subseteq \mathbf{P}) \Rightarrow 0 \in \mathrm{Spec}(\mathcal{Q}(A))$. Similarly, it is can be shown that $(A \in \mathbb{R}^{n \times n}, \mathcal{Q}(A) \not\subseteq \mathbf{P_0}), \Rightarrow \mathrm{Spec}(\mathcal{Q}(A))$ intersects the interior of $F(n)$.

Beyond qualitative classes

12. Many matrix-sets other than qualitative classes arise in applications. E.g.

   (a) (Assume AB is valid): $A\mathcal{Q}(B), \mathcal{Q}(A)\mathcal{Q}(B)$ (may/may not be a subset of a qualitative class).

   (b) Given $A \in \mathbb{R}^{n \times n}$, define $\mathcal{Q}^k(A) = \{B^k \colon B \in \mathcal{Q}(A)\}$. [Note: $\mathcal{Q}^2(A) \subseteq \mathcal{Q}(A)\mathcal{Q}(A)$]

   (c) More generally, given a matrix set $\mathcal{A}$, define $\mathcal{A}^k = \{A^k \colon A \in \mathcal{A}\}$. [Note: $\mathcal{A}^2 \subseteq \mathcal{A}\mathcal{A}$]

13. **Example.** Suppose $\mathcal{A}^2 \subseteq \mathbf{P}$. Then $\mathrm{Spec}\,\mathcal{A} \subseteq \mathbb{C}\backslash\sqrt{F(n)}$. In particular, $\mathrm{Spec}\,\mathcal{A}$ misses the imaginary axis. If A is sign-symmetric, then $A^2 \in \mathbf{P_0}$ and $\mathrm{Spec}\,\mathcal{A}$ misses the nonzero imaginary axis. [Sign-symmetric: $A[\alpha|\beta]A[\beta|\alpha] \geq 0, \forall \, \alpha, \beta$, i.e., no pair of oppositely placed minors has negative product.]

14. **Example.** The sets $A\mathcal{Q}(A^t)$ arise in the study of systems of chemical reactions. There are now a variety of necessary and sufficient conditions on A for $A\mathcal{Q}(A^t) \subseteq \mathbf{P_0}$, and a variety of other theory allowing claims about these sets.

Generalising the basic sign nonsingularity result

15. Given matrices A, B such that AB is square we define the signed bipartite digraph $G_{A,B}$ via

$$A(G_{A,B}) = \begin{pmatrix} \mathbf{0} & \mathrm{sign}\,A \\ \mathrm{sign}\,B & \mathbf{0} \end{pmatrix}.$$

16. Note that G_{A,A^t} can be seen as a signed undirected graph, the **bipartite graph of** A.

17. This generalises to longer products, e.g. $G_{A,B,C}$ is defined via

$$A(G_{A,B,C}) = \begin{pmatrix} \mathbf{0} & \mathrm{sign}\,A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{sign}\,B \\ \mathrm{sign}\,C & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

18. We **generalise the notion of odd**: a circuit C in $G_{A_1,\ldots,A_k}$ is "k-odd" if $|C|/k +$ (number of $-$ve arcs in C) is odd (note: $|C| = 0 \mod k$). $G_{A_1,A_2,\ldots,A_k} \in k\text{-}\mathbf{ODD}$ means all circuits in $G_{A_1,A_2,\ldots,A_k}$ are $k\text{-}\mathbf{ODD}$. We have [2]:

$$\boxed{G_{A_1,A_2,\ldots,A_k} \in k\text{-}\mathbf{ODD} \Leftrightarrow \mathcal{Q}(A_1)\mathcal{Q}(A_2)\cdots\mathcal{Q}(A_k) \subseteq \mathbf{P_0}}$$

19. **Remark.** For some computations, arcs in these digraphs may also be labelled with the absolute value of their associated matrix entry.

Exterior algebra

20. The vector space $\Lambda^k \mathbb{R}^n$ (the kth exterior power of $\mathbb{R}^n$) comprises finite formal linear combinations of elements of the form
$$u_1 \wedge u_2 \wedge \ldots \wedge u_k$$
where $u_i \in \mathbb{R}^n$, and "$\wedge$" is the wedge-product (a multilinear, alternating, product). A basis for $\mathbb{R}^n$ naturally determines a basis for $\Lambda^k \mathbb{R}^n$. $A \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$ determines transformations from $\Lambda^k \mathbb{R}^n$ to $\Lambda^k \mathbb{R}^n$:

   (a) $A^{(k)} : \Lambda^k \mathbb{R}^n \to \Lambda^k \mathbb{R}^n$ defined by $A^{(k)}(u_1 \wedge \cdots \wedge u_k) = (Au_1) \wedge \cdots \wedge (Au_k)$. [The "$k$th exterior power" or "$k$th multiplicative compound" of A].

   (b) $A^{[k]} : \Lambda^k \mathbb{R}^n \to \Lambda^k \mathbb{R}^n$ defined by $A^{[k]}(u_1 \wedge \cdots \wedge u_k) = \sum_{i=1}^{k} u_1 \wedge \cdots (Au_i) \wedge \cdots \wedge u_k$. [The "$k$th additive compound" of A.]

21. **Remark.** We can abbreviate some statements using compound matrices. E.g., A is a P-matrix iff $(A^{(k)})_{ii} > 0$ for all i, k; A is a P_0-matrix iff $(A^{(k)})_{ii} \geq 0$ for all i, k; A is sign-symmetric if $(A^{(k)})_{ij}(A^{(k)})_{ji} \geq 0$ for all i, j, k. The latter is a sufficient condition for $A^2 \in \mathbf{P_0}$, since $(A^2)^{(k)} = A^{(k)}A^{(k)}$ and so $[(A^2)^{(k)}]_{ii} = \sum_{\ell}(A^{(k)})_{i\ell}(A^{(k)})_{\ell i}$.

22. If $\operatorname{Spec} A = \{\lambda_1, \ldots, \lambda_n\}$ (a multiset).

   (a) $\operatorname{Spec} A^{(k)} = \{\prod_i \lambda_{\sigma_i}\}_{\sigma \subseteq \{1,\ldots,n\}, |\sigma|=k}$. (All k-fold products of distinct eigenvalues.)

   (b) $\operatorname{Spec} A^{[k]} = \{\sum_i \lambda_{\sigma_i}\}_{\sigma \subseteq \{1,\ldots,n\}, |\sigma|=k}$. (All k-fold sums of distinct eigenvalues.)

23. By analogy with $\mathcal{Q}^k(A) = \{B^k : B \in \mathcal{Q}(A)\}$, we have $\mathcal{Q}^{(k)}(A) = \{B^{(k)} : B \in \mathcal{Q}(A)\}$, etc.

24. **Example.** Given $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ define $\mathcal{A}^{[k]} = \{A^{[k]} : A \in \mathcal{A}\}$. What would $\mathcal{A}^{[2]} \subseteq \mathbf{P}$ (resp. $\mathcal{A}^{[2]} \subseteq \mathbf{P_0}$) imply about $\operatorname{Spec} \mathcal{A}$? For $i \neq j$, $\lambda_i + \lambda_j \notin F(n)$ (resp. $\lambda_i + \lambda_j \notin \operatorname{int} F(n)$). In particular there are no nonreal eigenvalues in the closed (resp. open) left half-plane of $\mathbb{C}$. What would $\mathcal{A}, \mathcal{A}^{[2]} \subseteq \mathbf{P}$ (resp. $\mathcal{A}, \mathcal{A}^{[2]} \subseteq \mathbf{P_0}$) imply? $\mathcal{A}$ is positive stable (resp. positive semistable).

Graphs from the second additive compound of a product [3]

25. Given $A, B^t \in \mathbb{R}^{n \times m}$ define the signed bipartite digraph on $\binom{n}{2} \times nm$ vertices $G_{A,B}^{[2]}$ as follows. One vertex set consists of all pairs $(i, j) \in \{1, \ldots, n\}^2$ with $i < j$ (represented as ij); one consists of all pairs $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ (represented as i^j). Arc (k^ℓ, ij) exists if $\ell \in \{i, j\}$ and $A_{\{i,j\}\backslash\ell, k} \neq 0$; arc (ij, k^ℓ) exists if $\ell \in \{i, j\}$ and $B_{k, \{i,j\}\backslash\ell} \neq 0$. Thus each arc is associated with an entry in either A or B. Arc (k^ℓ, ij) or (ij, k^ℓ) takes (resp. reverses) the sign of its associated matrix entry if $\ell = i$ (resp. $\ell = j$). Arcs may be labelled with the absolute value of their associated matrix entry. Oppositely directed pairs of arcs of the same sign are generally merged into a single undirected edge.

26. As before, the special case $G^{[2]}_{A,A^t}$ can be regarded as a signed bipartite graph.

27. **Example:** if $A = (1, 1, 1, 1)^t$, then $G^{[2]}_{A,A^t}$ is:

28. The key point is this: if $G^{[2]}_{A,B} \in$ 2-**ODD** then $(AB)^{[2]} \in \mathbf{P_0}$. For example, consider the sign pattern A and $G^{[2]}_{A,I}$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$G^{[2]}_{A,I} \in$ 2-**ODD** so $\mathcal{Q}^{[2]}(A) \subseteq \mathbf{P_0}$, so $\operatorname{Spec} \mathcal{Q}(A) \subseteq \{x + iy \in \mathbb{C} : y = 0 \text{ or } x \geq 0\}$. [Incidentally $G_{A,I} \notin$ 2-**ODD** and A is not necessarily positive semistable.]

29. Certain properties of $G^{[2]}_{A,B}$ beyond the parity of cycles can be important. For example, if G_{C,C^t} is acyclic, then $G^{[2]}_{C,C^t}$ has a property which implies for all $A, B^t \in \overline{\mathcal{Q}(C)}$: $AB, (AB)^{[2]} \in \mathbf{P_0}$, and so AB is positive semistable.

30. There are many other results using $G^{[2]}_{A,B}$, some involving computations on edge-labels, and leading to conclusions about the spectrum of the product AB.

31. **Conclusions.** This talk just scratches the surface of combinatorial approaches to claims about matrix spectra.

References

[1] R. B. Kellogg. On complex eigenvalues of M and P matrices. *Numer. Math.*, 19:70–175, 1972.

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[3] D. Angeli, M. Banaji, and C. Pantea. Combinatorial approaches to Hopf bifurcations in systems of interacting elements. http://arxiv.org/abs/1301.7076.