Solutions to Exercises
Chapter 1: What is Combinatorics?

1 For $n = 3,4,5$, calculate the number of ways of putting n letters into their envelopes so that every letter is incorrectly addressed. Calculate the ratio of this number to $n!$ in each case.

The numbers and ratios are as follows:

$n = 3$	2	$0.3333\ldots$
$n = 4$	9	0.375
$n = 5$	44	$0.3666\ldots$

Note that $1/\mathrm{e} = 0.367879\ldots$

(If you are unfamiliar with permutations, you can do this by listing all possibilities, though it gets a bit tedious. A slightly more refined analysis would go like this. For $n = 3$, the only possibility is "cyclic": A gets B's letter, B gets C's, and C gets A's, or the reverse of this. For $n = 4$, there are six "cyclic" solutions of this form, and another three where the letters are swapped in two pairs (for example, A and B get each other's letters, as do C and D). For $n = 5$, again either cyclic (24 cases) or a group of 2 and a group of 3 (20 cases).

The labour of the method increases rapidly with n. For $n = 6$, there are four cases to consider. On pages 57–59 and 77–78, you will find two different arguments which are valid for all n.

You can find more terms of the sequence here.

2 Solve Kirkman's problem for nine schoolgirls, walking for four days.

Look ahead to page 109 for a solution. The fifteen schoolgirls are on page 119.

You can read more about Kirkman's schoolgirls here.

3 Solve Euler's problem for nine, sixteen and twenty-five officers. Show that no solution is possible for four officers.

If capital letters denote ranks, and lower case letters regiments, one possible solution for each case is:

Aa	Bb	Cc
Bc	Ca	Ab
Cb	Ac	Ba

Aa	Bb	Cc	Dd
Bc	Ad	Da	Cb
Cd	Dc	Ab	Ba
Db	Ca	Bd	Ac

Aa	Bb	Cc	Dd	Ee
Bc	Cd	De	Ea	Ab
Ce	Da	Eb	Ac	Bd
Db	Ec	Ad	Be	Ca
Ed	Ae	Ba	Cb	Dc

With four officers we can suppose that we start with Aa in the first row and first column. But then the other officer in the same row, and the other officer in the same column, would both have to be Bb.

The fact that a solution exists for n^2 officers whenever $n \not\equiv 2 \pmod 4$ is proved on page 97.

You can read more about Euler's 36 officers here.

4 Test the assertion that the Ramsey game cannot end in a draw by playing it with a friend. Try to develop heuristic rules for successful play.

I don't know a winning strategy for this game. One has been worked out by computer, but is probably too complicated for mere mortals. A friend who tried it out suggested the following. *If you can't see what to do, play so as to make the position more complicated for your opponent to analyse.*

The proof that a draw is impossible is on page 149.

You can play the game online here.