7 Klein correspondence and triality

The orthogonal groups in dimension up to 8 have some remarkable properties. These include, in the finite case,

(a) isomorphisms between classical groups:

- $\mathrm{P}\Omega^{-}(4,q) \cong \mathrm{PSL}(2,q^2)$,
- $\mathrm{P}\Omega(5,q) \cong \mathrm{PSp}(4,q)$,
- $\mathrm{P}\Omega^{+}(6,q) \cong \mathrm{PSL}(4,q)$,
- $\mathrm{P}\Omega^{-}(6,q) \cong \mathrm{PSU}(4,q)$;

(b) unexpected outer automorphisms of classical groups:

- an automorphism of order 2 of $\mathrm{PSp}(4,q)$ for q even,
- an automorphism of order 3 of $\mathrm{P}\Omega^{+}(8,q)$;

(c) further simple groups:

- Suzuki groups;
- the groups $G_2(q)$ and ${}^3D_4(q)$;
- Ree groups.

In this section, we look at the geometric algebra underlying some of these phenomena.

Notation: we use $\mathrm{O}^{+}(2mF)$ for the isometry group of the quadratic form of Witt index m on a vector space of rank $2m$ (extending the notation over finite fields introduced earlier). We call this quadratic form Q *hyperbolic*. Moreover, the flat subspaces of rank 1 for Q are certain points in the corresponding projective space $\mathrm{PG}(2m-1,F)$; the set of such points is called a *hyperbolic quadric* in $\mathrm{PG}(2m-1,F)$.

We also denote the orthogonal group of the quadratic form

$$Q(x_1,\ldots,x_{2m+1}) = x_1x_2 + \cdots + x_{2m-1}x_{2m} + x_{2m+1}^2$$

by $\mathrm{O}(2m+1,F)$, again in agreement with the finite case.

7.1 Klein correspondence

The *Klein correspondence* relates the geometry of the vector space $V = F^4$ of rank 4 over a field F with that of a vector space of rank 6 over F carrying a quadratic form with Witt index 3.

It works as follows. Let W be the space of all 4×4 skew-symmetric matrices over F. Then W has rank 6: the above-diagonal elements of such a matrix may be chosen freely, and then the matrix is determined.

On the vector space W, there is a quadratic form Q given by

$$Q(X) = \mathrm{Pf}(X) \quad \text{for all } X \in W.$$

Recall the Pfaffian from Section 4.1, where we observed in particular that, if $X = (x_{ij})$, then

$$\mathrm{Pf}(X) = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}.$$

In particular, W is the sum of three hyperbolic planes, and the Witt index of Q is 3. There is an action ρ of $\mathrm{GL}(4,F)$ on W given by the rule

$$\rho(P) : X \mapsto P^\top XP$$

for $P \in \mathrm{GL}(4,F)$, $X \in W$. Now

$$\mathrm{Pf}(PXP^\top) = \det(P)\,\mathrm{Pf}(X),$$

so $\rho(P)$ preserves Q if and only if $\det(P) = 1$. Thus $\rho(\mathrm{SL}(4,F)) \leq \mathrm{O}(Q)$, and since $\mathrm{SL}(4,F)$ is equal to its derived group we have $\rho(\mathrm{SL}(4,F)) \leq \Omega^+(6,F)$. It is easily checked that the kernel of ρ consists of scalars; so in fact we have $\mathrm{PSL}(4,F) \leq \mathrm{P}\Omega^+(6,F)$.

A calculation shows that in fact equality holds here. (More on this later.)

Theorem 7.1 $\mathrm{P}\Omega^+(6,F) \cong \mathrm{PSL}(4,F)$. ∎

Examining the geometry more closely throws more light on the situation. Since the Pfaffian is the square root of the determinant, we have

$$Q(X) = 0 \text{ if and only if } X \text{ is singular.}$$

Now a skew-symmetric matrix has even rank; so if $Q(X) = 0$ but $X \neq 0$, then X has rank 2.

Exercise 7.1 Any skew-symmetric $n \times n$ matrix of rank 2 has the form

$$X(v,w) = v^\top w - w^\top v$$

for some $v,w \in F^n$.

Hint: Let B be such a matrix and let v and w span the row space of B. Then $B = x^\top v + y^\top w$ for some vectors x and y. Now by transposition we see that $\langle x,y\rangle = \langle v,w\rangle$. Express x and y in terms of v and w, and use the skew-symmetry to determine the coefficients up to a scalar factor.

Now $X(v,w) \neq 0$ if and only if v and w are linearly independent. If this holds, then the row space is spanned by v and w. Moreover,

$$X(av + cw, bc + dw) = (ad - bc)X(v,w).$$

So there is a bijection between the rank 2 subspaces of F^4 and the flat subspaces of W of rank 1. In terms of projective geometry, we have:

Proposition 7.2 *There is a bijection between the lines of* $\mathrm{PG}(3,F)$ *and the points on the hyperbolic quadric in* $\mathrm{PG}(5,F)$*, which intertwines the natural actions of* $\mathrm{PSL}(4,F)$ *and* $\mathrm{P}\Omega^+(6,F)$. ∎

This correspondence is called the *Klein correspondence*, and the quadric is often referred to as the *Klein quadric*.

Now let A be a non-singular skew-symmetric matrix. The stabiliser of A in $\rho(\mathrm{SL}(4,F))$ consists of all matrices P such that $PAP^\top = A$. These matrices comprise the symplectic group (see the exercise below). On the other hand, A is a vector of W with $Q(A) \neq 0$, and so the stabiliser of A in the orthogonal group is the 5-dimensional orthogonal group on $A^\perp$ (where orthogonality is with respect to the bilinear form obtained by polarising Q). Thus, we have

Theorem 7.3 $\mathrm{P}\Omega(5,F) \cong \mathrm{PSp}(4,F)$. ∎

Exercise 7.2 Let A be a non-singular skew-symmetric 4×4 matrix over a field F. Prove that the following assertions are equivalent, for any vectors $v,w \in F^4$:

(a) $X(v,w) = v^\top w - w^\top v$ is orthogonal to A, with respect to the bilinear form obtained by polarising the quadratic form $Q(X) = \mathrm{Pf}(X)$;

(b) v and w are orthogonal with respect to the symplectic form with matrix $A^\dagger$, that is, $vA^\dagger w^\top = 0$.

Here the matrices A and $A^\dagger$ are given by

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}, \qquad A^\dagger = \begin{pmatrix} 0 & a_{34} & -a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & -a_{13} \\ a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & a_{13} & -a_{12} & 0 \end{pmatrix}.$$

Now show that the transformation induced on W by a 4×4 matrix P fixes A if and only if $PA^\dagger P^\top = A^\dagger$, in other words, P is symplectic with respect to $A^\dagger$.

Note that, if A is the matrix of the standard symplectic form, then so is $A^\dagger$.

Now, we have two isomorphisms connecting the groups $\mathrm{PSp}(4,F)$ and $\mathrm{P}\Omega(5,F)$ in the case where F is a perfect field of characteristic 2. We can apply one and then the inverse of the other to obtain an automorphism of the group $\mathrm{PSp}(4,F)$. Now we show geometrically that it must be an outer automorphism.

The isomorphism in the preceding section was based on taking a vector space of rank 5 and factoring out the radical Z. Recall that, on any coset $Z+u$, the quadratic form takes each value in F precisely once; in particular, there is a unique vector in each coset on which the quadratic form vanishes. Hence there is a bijection between vectors in F^4 and vectors in F^5 on which the quadratic form vanishes. This bijection is preserved by the isomorphism. Hence, under this isomorphism, the stabiliser of a point of the symplectic polar space is mapped to the stabiliser of a point of the orthogonal polar space.

Now consider the isomorphism given by the Klein correspondence. Points on the Klein quadric correspond to lines of $\mathrm{PG}(3,F)$, and it can be shown that, given a non-singular matrix A, points of the Klein quadric orthogonal to A correspond to flat lines with respect to the corresponding symplectic form on F^4. In other words, the isomorphism takes the stabiliser of a line (in the symplectic space) to the stabiliser of a point (in the orthogonal space).

So the composition of one isomorphism with the inverse of the other interchanges the stabilisers of points and lines of the symplectic space, and so is an outer automorphism of $\mathrm{PSp}(4,F)$.

7.2 The Suzuki groups

In certain cases, we can choose the outer automorphism such that its square is the identity. Here is a brief account.

Theorem 7.4 *Let F be a perfect field of characteristic 2. Then the polar space defined by a symplectic form on F^4 itself has a polarity if and only if F has an automorphism σ satisfying $\sigma^2 = 2$, where 2 denotes the automorphism $x \mapsto x^2$ of F.*

Proof We take the standard symplectic form

$$B((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) = x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3.$$

The Klein correspondence takes the line spanned by the two points (x_1,x_2,x_3,x_4) and (y_1,y_2,y_3,y_4) to the point wit coordinates z_{ij}, for $1 \le i < j \le 4$, where $z_{ij} = x_iy_j + x_jy_i$. This point lies on the Klein quadric with equation

$$z_{12}z_{34} + z_{13}z_{24} + z_{14}z_{23} = 0,$$

and also (if the line is flat) on the hyperplane $z_{12} + z_{34} = 0$. This hyperplane is orthogonal to the point p with $z_{12} = z_{34} = 1$, $z_{ij} = 0$ otherwise. Using coordinates $(z_{13},z_{24},z_{14},z_{23})$ in $p^\perp/p$, we obtain a point of the symplectic space representing the line. This gives the duality δ previously defined.

Now take a point $q = (a_1,a_2,a_3,a_4)$ of the original space, and calculate its image under the duality, by choosing two flat lines through q, calculating their images, and taking the line joining them. Assuming that a_1 and a_4 are non-zero, we can use the lines joining q to the points $(a_1,a_2,0,0)$ and $(0,a_4,a_1,0)$; their images are $(a_1a_3,a_2a_4,a_1a_4,a_2a_3)$ and $(a_1^2,a_4^2,0,a_1a_2+a_3a_4)$. Now compute the image of the line joining these two points, which turns out to be $(a_1^2,a_2^2,a_3^2,a_4^2)$. In all other cases, the result is the same. So $\delta^2 = 2$.

If there is a field automorphism σ such that $\sigma^2 = 2$, then $\delta\sigma^{-1}$ is a duality whose square is the identity, that is, a polarity.

Conversely, suppose that there is a polarity τ. Then $\delta\tau$ is a collineation, hence a product of a linear transformation and a field automorphism, say $\delta\tau = g\sigma$. Since $\delta^2 = 2$ and $\tau^2 = 1$, we have that $\sigma^2 = 2$ as required. ∎

It can further be shown that the set of collineations which commute with this polarity is a group G which acts doubly transitively on the set Ω of absolute points of the polarity, and that Ω is an *ovoid* (that is, each flat line contains a unique point of Ω. If $|F| > 2$, then the derived group of G is a simple group, the *Suzuki group* $\mathrm{Sz}(F)$.

The finite field $\mathrm{GF}(q)$, where $q = 2^m$, has an automorphism σ satisfying $\sigma^2 = 2$ if and only if m is odd (in which case, $2^{(m+1)/2}$ is the required automorphism). In this case we have $|\Omega| = q^2+1$, and $|\mathrm{Sz}(q)| = (q^2+1)q^2(q-1)$. For $q = 2$, the Suzuki group is not simple, being isomorphic to the Frobenius group of order 20.

7.3 Clifford algebras and spinors

We saw earlier (Proposition 3.11) that, if Q is a hyperbolic quadratic form on F^{2m}, then the maximal flat subspaces for Q fall into two families $\mathcal{S}^+$ and $\mathcal{S}^-$, such that if S and T are maximal flat subspaces, then $S \cap T$ has even codimension in S and T if and only if S and T belong to the same family.

In this section we represent the maximal flat subspaces as points in a larger projective space, based on the space of *spinors*. The construction is algebraic. First we briefly review facts about multilinear algebra.

Let V be a vector space over a field F, with rank m. The *tensor algebra* of V, written $\bigotimes V$, is the largest associative algebra generated by V in a linear fashion. In other words,

$$\bigotimes V = \bigoplus_{n \geq 0} \bigotimes^{n} V,$$

where, for example, $\bigotimes^2 V = V \otimes V$ is spanned by symbols $v \otimes w$, with $v, w \in V$, subject to the relations

$$\begin{aligned} (v_1 + v_2) \otimes w \quad &= \quad v_1 \otimes w + v_2 \otimes w, \\ v \otimes (w_1 + w_2) \quad &= \quad v \otimes w_1 + v \otimes w_2, \\ (cv) \otimes w = \quad c(v \otimes w) \quad &= v \otimes cw. \end{aligned}$$

(Formally, it is the quotient of the free associative algebra over F with basis V by the ideal generated by the differences of the left and right sides of the above identities.) The algebra is $\mathbb{N}$-graded, that is, it is a direct sum of components $V_n = \bigotimes^n V$ indexed by the natural numbers, and $V_{n_1} \otimes V_{n_2} \subseteq V_{n_1+n_2}$.

If $(e_1, \ldots, e_m)$ is a basis for V, then a basis for $\bigotimes^n V$ consists of all symbols

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n},$$

for $i_1, \ldots, i_n \in \{1, \ldots, m\}$; thus,

$$\mathrm{rk}(\bigotimes^{n} V) = m^n.$$

The *exterior algebra* of V is similarly defined, but we add an additional condition, namely $v \wedge v = 0$ for all $v \in V$. (In this algebra we write the multiplication as $\wedge$.) Thus, the exterior algebra $\bigwedge V$ is the quotient of $\bigotimes V$ by the ideal generated by $v \otimes v$ for all $v \in V$.

In the exterior algebra, we have $v \wedge w = -w \wedge v$. For

$$0 = (v+w) \wedge (v+w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w,$$

and the first and fourth terms on the right are zero. This means that, in any expression $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, we can rearrange the factors (possibly changing the signs), and if two adjacent factors are equal then the product is zero. Thus, the nth component $\bigwedge^n V$ has a basis consisting of all symbols

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$$

where $i_1 < i_2 < \ldots < i_n$. In particular,

$$\mathrm{rk}(\bigwedge^n V = \binom{m}{n},$$

so that $\bigwedge^n V + \{0\}$ for $n > m$; and

$$\mathrm{rk}(\bigwedge V) = \sum_{n=0}^{m} \binom{m}{n} = 2^m.$$

Note that $\mathrm{rk}(\bigwedge^m V) = 1$. Any linear transformation T of V induces in a natural way a linear transformation on $\bigotimes^n V$ or $\bigwedge^n V$ for any n. In particular, the transformation $\bigwedge^m T$ induced on $\bigwedge^m V$ is a scalar, and this provides a coordinate-free definition of the determinant:

$$\det(T) = \bigwedge^m T.$$

Now let Q be a quadratic form on V. We define the *Clifford algebra* $C(Q)$ of Q to be the largest associative algebra generated by V in which the relation

$$v \cdot v = Q(v)$$

holds. (We use $\cdot$ for the multiplication in $C(Q)$. Note that, if Q is the zero form, then $C(Q)$ is just the exterior algebra. If B is the bilinear form obtained by polarising Q, then we have

$$v \cdot w + w \cdot v = B(v, w).$$

This follows because

$$Q(v+w) = (v+w) \cdot (v+w) = v \cdot v + v \cdot w + w \cdot v + w \cdot w$$

and also

$$Q(v+w) = Q(v) + Q(w) + B(v,w).$$

Now, when we arrange the factors in an expression like

$$e_{i_1} \cdot e_{i_2} \cdots e_{i_n},$$

we obtain terms of degree $n-2$ (and hence $n-4, n-6, \ldots$ as we continue). So again we can say that the nth component has a basis consisting of all expressions

$$e_{i_1} \cdot e_{i_2} \cdots e_{i_n},$$

where $i_1 < i_2 < \ldots < i_n$, so that $\mathrm{rk}(C(Q)) = 2^m$. But this time the algebra is not graded but only $\mathbb{Z}_2$-graded. That is, if we let C^0 and C_1 be the sums of the components of even (resp. odd) degree, then $C^i \cdot C^j \subseteq C^{i+j}$, where the superscripts are taken modulo 2.

Suppose that Q polarises to a non-degenerate bilinear form B. Let $G = \mathrm{O}(Q)$ and $C = C(Q)$. The *Clifford group* $\Gamma(Q)$ is defined to be the group of all those units $s \in C$ such that $s^{-1}Vs = V$. Note that $\Gamma(Q)$ has an action χ on V by the rule

$$s : v \mapsto s^{-1}vs.$$

Proposition 7.5 *The action χ of $\Gamma(Q)$ on V is orthogonal.*

Proof

$$Q(s^{-1}vs) = (s^{-1}vs)^2 = s^{-1}v^2s = s^{-1}Q(v)s = Q(v),$$

since $Q(v)$, being a scalar, lies in the centre of C. ■

We state without proof:

Proposition 7.6 *(a)* $\chi(\Gamma(Q)) = \mathrm{O}(Q)$*;*

(b) $\ker(\chi)$ *is the multiplicative group of invertible central elements of* $C(Q)$*.* ■

The structure of $C(Q)$ can be calculated in special cases. The one which is of interest to us is the following:

Theorem 7.7 *Let Q be hyperbolic on F^{2m}. Then $C(Q) \cong \mathrm{End}(S)$, where S is a vector space of rank 2^m over F called the* space of spinors*. In particular, $\Gamma(Q)$ has a representation on S, called the* spin representation*.*

The theorem follows immediately by induction from the following lemma:

Lemma 7.8 *Suppose that $Q = Q' + yz$, where y and z are variables not occurring in Q'. Then $C(Q) \cong M_{2\times 2}(C(Q'))$.*

Proof Let $V = V' \perp \langle e, f\rangle$. Then V' generates $C(Q')$, and V', e, f generate $C(Q)$. We represent these generators by 2×2 matrices over $C(Q')$ as follows:

$$\begin{aligned} v &\mapsto \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}, \\ e &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ f &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Some checking is needed to establish the relations. ■

Let S be the vector space affording the spin representation. If U is a flat m-subspace of V, let f_U be the product of the elements in a basis of U. (Note that f_U is uniquely determined up to multiplication by non-zero scalars; indeed, the subalgebra of $C(Q)$ generated by U is isomorphic to the exterior algebra of U.) Now it can be shown that Cf_U and f_UC are minimal left and right ideals of C. Since $C \cong \mathrm{End}(S)$, each minimal left ideal has the form $\{T : VT \subseteq X\}$ and each minimal right ideal has the form $\{T : \ker(T) \supseteq Y\}$, where X and Y are subspaces of V of dimension and codimension 1 respectively. In particular, a minimal left ideal and a minimal right ideal intersect in a subspace of rank 1.

Thus we have a map σ from the set of flat m-subspaces of V into the set of 1-subspaces of S.

Vectors which span subspaces in the image of σ are called *pure spinors*.

Theorem 7.9 *$S = S^+ \oplus S^-$, where $\mathrm{rk}(S^+) = \mathrm{rk}(S^-) = 2^{m-1}$. Moreover, any pure spinor lies in either S^+ or S^- according as the corresponding maximal flat subspace lies in $\mathcal{S}^+$ or $\mathcal{S}^-$.* ■

Furthermore, it is possible to define a quadratic form γ on S, whose corresponding bilinear form β is non-degenerate, so that the following holds:

- if m is odd, then S^+ and S^- are flat subspaces for γ, and β induces a non-degenerate pairing between them;

- if m is even, then S^+ and S^- are orthogonal with respect to β, and γ is non-degenerate hyperbolic on each of them.

We now look briefly at the case $m = 3$. In this case, $\mathrm{rk}S^+ = \mathrm{rk}(S^-) = 4$. The Clifford group has a subgroup of index 2 fixing S^+ and S^-, and inducing dual representations of $\mathrm{SL}(4,F)$ on them. We have here the Klein correspondence in another form.

This case $m = 4$ is even more interesting, as we see in the next section.

7.4 Triality

Suppose that, in the notation of the preceding section, $m = 4$. That is, Q is a hyperbolic quadratic form on $V = F^8$, and the spinor space S is the direct sum of two subspaces S^+ and S^- of rank 8, each carrying a hyperbolic quadratic form of rank 8. So each of these two spaces is isomorphic to the original space V. There is an isomorphism τ (the *triality map*) of order 3 which takes V to S^+ to S^- to V, and takes Q to $\gamma|S^+$ to $\gamma|S^-$ to Q. Moreover, τ induces an outer automorphism of order 3 of the group $\mathrm{P}\Omega^+(8,F)$.

Moreover, we have:

Proposition 7.10 *A vector $s \in S$ is a pure spinor if and only if*

(a) $s \in S^+$ *or* $s \in S^-$*; and*

(b) $\gamma(s) = 0$. ∎

Hence τ takes the stabiliser of a point to the stabiliser of a maximal flat subspace in $\mathcal{S}^+$ to the stabiliser of a maximal flat subspace in $\mathcal{S}^-$ back to the stabiliser of a point.

It can be shown that the centraliser of τ in the orthogonal group is the group $G_2(F)$, an *exceptional group of Lie type*, which is the automorphism group of an octonion algebra over F.

Further references for this chapter are in C. Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras* (Collected Works Vol. 2), Springer, 1997.