Synchronization 8: The infinite

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Synchronization in the infinite case

We cannot simply take the definition of a synchronizing finite permutation group and extend it to the infinite: there would be no such groups!

Let Ω be an infinite set. Then both the injective maps, and the surjective maps, on Ω form submonoids of the full transformation monoid; they contain the symmetric group but no reset word.

Since the essence of synchronization seems to involve mapping different states to the same place, it is reasonable to require that the map we adjoin is not injective.

Our first attempt at a suitable definition is based on the following fact about the finite case:

Theorem 1. *Let M be a transformation monoid on a finite set Ω. Suppose that, for any $v, w \in \Omega$, there exists $f \in M$ with $vf = wf$. Then M is synchronizing.*

Proof. Let f' be an element of M whose image S has least possible cardinality. If $|S| > 1$, choose distinct $v, w \in S$, and choose f as in the hypothesis; then the image of $f'f$ is strictly smaller than that of f. □

Accordingly, we could try a definition along the following lines:

- A transformation monoid M on Ω is *synchronizing* if, for any $v, w \in \Omega$, there exists $f \in M$ with $vf = wf$; equivalently, $\mathrm{Gr}(M)$ is the null graph on Ω.
- A permutation group G on Ω is *synchronizing* if, for any map $f : \Omega \to \Omega$ which is not injective, the monoid $\langle G, f \rangle$ is synchronizing.

Unfortunately this doesn't give anything interesting!

Ramsey's Theorem

Ramsey's Theorem is much more general than the form given here; but this is all we need.

Where necessary, we assume the Axiom of Choice, one of whose consequences is that an infinite set contains a countably infinite subset.

Theorem 2. *An infinite graph contains either an infinite clique or an infinite independent set.*

By our remark, it suffices to prove this for a countably infinite graph.

Proof. Let $v_1, v_2, \ldots$ be the vertices. We construct inductively a sequence of triples (x_i, Y_i, ϵ_i), where the x_i are distinct vertices, Y_i are infinite decreasing subsets of vertices, $x_i \in Y_j$ if and only if $j < i$, and x_i is joined to all or no vertices of Y_i according as $\epsilon_j = 1$ or $\epsilon_j = 0$. We begin with Y_0 the whole vertex set.

Choose $x_i \in Y_{i-1}$. By the Pigeonhole Principle, either x_i has infinitely many neighbours, or it has infinitely many non-neighbours, in Y_{i-1}; let Y_i be the appropriate infinite set and choose ϵ_i appropriately.

Now the sequence $(\epsilon_1, \epsilon_2, \ldots)$ has a constant subsequence; the points x_i corresponding to this subsequence form a clique or independent set, depending on the constant value of ϵ_i. ☐

We use Ramsey's Theorem to show that the notion of "synchronizing" we just defined is not interesting, at least for permutation groups of countable degree.

Theorem 3. *Let G be a permutation group of countable degree. Then G is synchronizing if and only if it is* 2*-set transitive.*

Proof. Suppose that G is not 2-set transitive. Then there is a non-trivial G-invariant graph X (take a G-orbit on 2-sets as edges). Replacing X by its complement if necessary, and using Ramsey's theorem, we may assume that X has a countable clique Y.

Let v and w be non-adjacent vertices. Choose a bijection f from $\Omega \setminus w$ to Y, and extend it by setting $f(w) = f(v)$. Clearly f is an endomorphism of X collapsing v and w, and $\langle G, f\rangle$ is not a synchronizing monoid.

Conversely, if G is 2-set transitive and f a map satisfying $vf = wf$, then $(vg)(g^{-1}f) = (wg)(g^{-1}f)$ for any $g \in G$; so $\langle G, f\rangle$ collapses all pairs, and G is synchronizing. ☐

Weak synchronization

We look at a couple of modifications. We say that G is *weakly synchronizing* if, for any map $f : \Omega \to \Omega$ of finite rank (that is, having finite image), the monoid $\langle G, f\rangle$ contains a reset word.

Now imprimitive groups may be weakly synchronizing; but it is true that a weakly synchronizing group cannot have a finite system of blocks of imprimitivity.

For if S is a transversal for such a system, and f is the map taking any point of Ω to the representative point of f, then $\langle G, f\rangle$ contains no reset word.

Note also that, if M is a transformation monoid containing an element of finite rank, and $\mathrm{Gr}(M)$ is complete, then M contains a reset word.

Strong synchronization

Another possible approach: since, in general, words in $\langle G, f\rangle$ will not be reset words, we should allow infinite words. This requires some preliminary thought.

Let M be a transformation monoid on Ω, and let $\overline{M}$ be its *closure* in the topology of pointwise convergence: a sequence (f_n) of element of M converges to the limit f if, for all $v \in \Omega$, there exists n_0 such that $vf_n = vf$ for all $n \geq n_0$.

Now we say that a permutation group G is *strongly synchronizing* if, for any map f which is not injective, the closure of $M = \langle G, f\rangle$ contains an element of rank 1.

Theorem 4.
- *A strongly synchronizing group is synchronizing.*
- *A* 2*-set transitive group of countable degree is strongly synchronizing.*

As a consequence of this theorem and the previous one about synchronizing groups, a permutation group of countable degree is strongly synchronizing if and only if it is 2-set transitive.

Proof. (a) Let f be a map which is not injective, and let (f_n) be a sequence of elements of $\langle G, f\rangle$ converging to a rank 1 function with image $\{z\}$, and choose two distinct points x and y. There exist n_1 and n_2 such that $xf_n = z$ for $n \geq n_1$ and $yf_n = z$ for $n \geq n_2$. So, if $n = \max(n_1, n_2)$, then $f_n \in \langle G, f\rangle$ and $xf_n = yf_n$. So G is synchronizing.

(b) Let G be 2-set transitive and let f be a function which is not injective. Choose two points x and y with $xf = yf$. By post-multiplication by an element of G, we can assume that $xf = x$.

Enumerate Ω, as $\{x_1, x_2, \ldots\}$, with $x_1 = x$, and construct a sequence (f_n) of elements of $\langle G, f\rangle$ as follows. Begin with $f_1 = f$. Now suppose that f_n is defined, and satisfies $x_m f_n = x$ for $m \leq n$. If $x_{n+1} f_n = x$, then choose $f_{n+1} = f_n$. Otherwise, choose $g \in G$ mapping $\{x, x_{n+1}\}$ to $\{x, y\}$, and let $f_{n+1} = f_n g f$. Clearly $x_m f_{n+1} = x$ for all $m \leq n+1$. So the sequence converges to the constant function with value x. ☐

Larger infinities

I know nothing about synchronization for larger infinite sets. But the proof that "synchronizing" is equivalent to "2-set transitive" fails, because of the failure of Ramsey's theorem to guarantee a clique or independent set of the same cardinality as Ω.

I do not know whether the two concepts are equivalent or not for sets of larger cardinalities. The answer might depend on the choice of set-theoretic axioms.

Example 5. The Axiom of Choice implies that there is a *well-ordering* of $\mathbb{R}$, a total ordering in which every non-empty subset has a least element. Choose such a well-ordering $\prec$. Now form a graph by joining v and w if $\prec$ and the usual order $<$ agree on $\{v, w\}$, and not if they disagree.

We claim that there is no uncountable clique. Let Y be a clique; then Y is well-ordered by the usual order on $\mathbb{R}$. In a well-order, each non-maximal element v has an immediate successor v'; choose a rational number $q(v)$ in the interval (v, v'). The chosen rationals are all distinct.

Reversing the usual order shows that the complementary graph has the same form; so the graph we constructed has no uncountable independent set either.

Hulls

The definition of cores in the infinite case is probematic, since it is not clear what "minimal" means. However, hulls can be defined as usual:

Let X be on the vertex set Ω. The *hull* of X is the graph $\mathrm{Gr}(\mathrm{End}(X))$; that is, two vertices v, w are joined in $\mathrm{Hull}(X)$ if and only if there is no endomorphism f of X satisfying $vf = wf$.

Theorem 6. *Any countable graph containing an infinite clique is a hull.*

This follows just as in our previous argument using Ramsey's theorem.

What happens for graphs with finite clique size?

Finite clique number

Here are two results on graphs with finite clique number.

Theorem 7. *Let X be a graph having an endomorphism of finite rank. Then the clique number and chromatic number of* $\mathrm{Hull}(X)$ *are equal (and finite).*

Proof. Without loss, X is a hull. Now if f is an endomorphism of minimum rank, then the image of f is a clique, and f is a proper colouring. □

The following result is due to Nick Gravin, a student of Dima Pasechnik.

Theorem 8. *Let X be an infinite hull with finite clique number. Then the chromatic number of X is equal to the clique number.*

Proof. Given a finite subgraph Y of X, if Y is not complete, then there is an endomorphism f_1 of X collapsing a non-edge of Y; if Yf_1 is not complete, there is an endomorphism f_2 collapsing a non-edge of Yf_1; and so on. We end with a homomorphism of Y to a complete graph of size at most $\omega(X)$. S0 $\chi(Y) \leq \omega(X)$ for any finite subgraph Y. A compactness argument shows that $\chi(X) \leq \omega(X)$, so equality holds. □