Resonances of Dynamical Systems and Fredholm-Riesz Operators on Rigged Hilbert Spaces

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Abstract

Resonances of dynamical systems are defined as the singularities of the analytically continued resolvent of the restriction of the Frobenius-Perron operator to suitable test-function spaces. A sufficient condition for resonances to arise from a meromorphic continuation to the entire plane is that the Frobenius-Perron operator is a Fredholm-Riesz operator on a rigged Hilbert space. After a discussion of spectral theory in locally convex topological vector spaces we illustrate the approach for a simple chaotic system, namely the Rényi map.

Keywords: Resonances, chaotic maps, Fredholm-Riesz operators, rigged Hilbert spaces

1 Introduction

In the study of chaotic dynamical systems time correlation functions play a central *rôle*, as they are generally the only experimentally accessible quantities. Ruelle [35] showed that the Fourier transform of the two-time correlation function, i.e. the power spectrum, of certain chaotic dynamical systems can be described by suitably chosen evolution operators of these systems. More precisely, complex poles of the meromorphic extension of the power spectrum, the so-called *Ruelle resonances*, are logarithms of eigenvalues of the Frobenius-Perron operator. Only their residues, but not their position, depend on the choice of observables (see also [10]). There is a *caveat* here: the location of the poles (or the isolated eigenvalues) is independent of the very choice of observables only if the 'level of observation' has been specified in advance. By this we mean that the class of observables (or the domain of definition of the Frobenius-Perron operator) has been adequately restricted. In the case of a piecewise expanding interval transformation, for example, Keller [20] showed that the L^2-spectrum of the associated Frobenius-Perron operator fills the unit disk, whereas the BV-spectrum, i.e. the spectrum of the restriction of the Frobenius-Perron operator to the space of functions of bounded variation, consists of two parts: eigenvalues filling the interior of a closed disk centred at the origin, which is strictly contained in the unit disk, and isolated eigenvalues of finite multiplicity outside this disk, but contained in the unit disk. This means that there are square-integrable functions whose power spectrum cannot be meromorphically continued to a strip containing the real line, whereas this is always the case for functions of bounded variation. In particular, a generic 'L^2-observation' will not have poles in its power spectrum, and if there is a pole it might not coincide with that of another generic L^2-observation. The choice of the test-function space on which the Frobenius-Perron operator is considered is therefore not purely academic.

Following [2] we understand resonances of dynamical systems as singularities of the analytically continued resolvent of the evolution operator on a rigged Hilbert space: given an evolution operator V on a Hilbert space $\mathcal{H}$, a subspace Φ of $\mathcal{H}$ — the so-called test-function space — is chosen, such that the following holds:

1) Φ carries a topology τ with respect to which it becomes a locally convex topological vector space;

2) (Φ, τ) is continuously and densely embedded into $\mathcal{H}$, i.e. the topology τ on Φ is stronger than the one induced by $\mathcal{H}$ and Φ is dense in $\mathcal{H}$;

3) (Φ, τ) is quasi-complete and barrelled

4) Φ is stable with respect to the adjoint $V^\dagger$ of V, i.e. $V^\dagger \Phi \subset \Phi$;

5) the adjoint $V^\dagger$ is continuous on (Φ, τ).

The triplet

$$\Phi \subset \mathcal{H} \subset \Phi',$$

where Φ' denotes the topological dual of Φ, is called a *rigged Hilbert space* or a *Gelfand triplet* (see [7, 14, 15, 12] for details). The operator V then has an extension V_{ext} to Φ' *qua* duality, which is defined by

$$(\phi|V_{\text{ext}}f) = (V^\dagger\phi|f)$$

for every $\phi \in \Phi$ and every $f \in \Phi'$. If the test-function space is suitably chosen, the resolvent of the restriction of $V^\dagger$ to Φ or the resolvent of the extension V_{ext} can be analytically continued into the Hilbert space spectrum of V giving rise to singularities, which are the resonances of the dynamical system. The choice of Φ and thus the concept of resonances depends on the physical observation of the system.

In the following we shall be concerned with discrete time dynamical systems or maps, i.e. endomorphisms S of a measure space X, for which the evolution operator V is given by the Koopman operator [24]:

$$(Vf)(x) = f(Sx),$$

where f is a square-integrable phase function. Its adjoint is the Frobenius-Perron operator. Our experience with chaotic maps so far indicates that the test-function spaces for resonances can be chosen such that the resolvent of the Frobenius-Perron operator can be meromorphically extended to the entire complex plane with the origin removed, i.e. the only resonances in the complex plane without the origin are poles. If in addition the eigenprojections (the residues) are finite rank operators then the operator is a *Fredholm-Riesz operator*. This is the case for a well-known example of a chaotic map, namely the Rényi map, for which there exist test-function spaces on which the associated Frobenius-Perron operator is a Fredholm-Riesz operator. Since most of these spaces are not normable, we briefly review the spectral theory of operators on locally convex topological vector spaces.

2 Spectral theory in locally convex topological vector spaces

The spectral theory of operators has its roots in the theory of matrices and the theory of integral equations, and, as such, goes back at least to the second half of the last century. Its present form grew out of a considerable abstraction of these ideas yielding a theory which is based on a fertile combination of function-theoretic and algebraic concepts. At the heart of this theory is the notion of the *spectrum*, which generalises the latent roots of a matrix and the singular values of an integral equation. Let us recall that a complex number λ is said to belong to the spectrum of a continuous operator T of a Banach space, if $\lambda - T$ fails to have a continuous inverse. With this definition the spectrum of T is a non-empty compact subset of the complex plane.

An extension of the above results to more general topological vector spaces relies heavily on a suitable concept of spectrum. If the very same definition is used for a continuous operator on a locally convex space, the spectrum may be empty or unbounded, and might not be closed (see [25] for examples). To our knowledge, the first concept of spectrum to provide a satisfactory analogy of the results in Banach spaces is due to Williamson [42] (unpublished). Not much later, Waelbroeck employed a similar idea to develop a spectral theory for locally convex algebras, which incorporates the main features of the Banach space theory [40, 41]. Parts of his theory were generalised by Neubauer [27], Allan [1], Moore [26], and König [21]. The only textbook to give a general account seems to be [16].

We shall briefly outline the most important elements of the spectral theory of operators on locally convex topological vector spaces leading to the Riesz decomposition theorem. The background and terminology may be found in [37] or [22, 23].

Let Φ be a barrelled quasi-complete locally convex topological vector space over $\mathbb{C}$. Let $\mathfrak{S}$ be a covering family of bounded subsets of Φ. $\mathcal{L}_{\mathfrak{S}}(E)$ denotes the space of continuous operators on Φ equipped with the topology of uniform convergence on sets belonging to $\mathfrak{S}$. Note that $\mathcal{L}_{\mathfrak{S}}(E)$ is quasi-complete by [37, III.4.4, Corollary]. If $\mathfrak{S}$ has the property that

$$V(B) \in \mathfrak{S} \text{ for } V \in \mathcal{L}(E), B \in \mathfrak{S},$$

then $\mathfrak{S}$ is called *admissible*, and $\mathcal{L}_{\mathfrak{S}}(E)$ becomes a locally convex algebra with separately continuous multiplication for admissible $\mathfrak{S}$. A function defined on an open subset $\Omega \subset \mathbb{C}$ with values in $\mathcal{L}_{\mathfrak{S}}(E)$ is called *holomorphic*, if for every $z_0 \in \Omega$ the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The collection of all functions which are holomorphic in Ω shall be denoted by $H(\Omega, \mathcal{L}_{\mathfrak{S}}(E))$. Since $\mathcal{L}_{\mathfrak{S}}(E)$ is locally convex and quasi-complete, the integral of functions in $H(\Omega, \mathcal{L}_{\mathfrak{S}}(E))$ over rectifiable Jordan curves lying in Ω may be defined as it is done for $\mathbb{C}$-valued functions. Moreover, it turns out that a theory of $\mathcal{L}_{\mathfrak{S}}(E)$-valued holomorphic functions can be developed as in the one-dimensional case. In particular, we get the analogues of Cauchy's [16, III.27] and Taylor's [16, III.33b] theorem, as well as Laurent series expansions [16, III.38] for functions holomorphic in an annulus. Thus, the notions 'pole' and 'residue' can be defined as usual.

Let us remark that Cauchy's integral theorem together with the uniform boundedness principle imply that $f \in H(\Omega, \mathcal{L}_{\mathfrak{S}}(E))$ for some admissible $\mathfrak{S}$, if and only if $f \in H(\Omega, \mathcal{L}_{\mathfrak{S}'}(E))$ for any admissible $\mathfrak{S}'$, i.e. the notion of holomorphicity does not depend on the choice of $\mathfrak{S}$.

We now turn to the definition of the spectrum of an operator $V \in \mathcal{L}(E)$. The *resolvent set* $\varrho(V)$ of V is the largest open subset Ω of the domain of definition of $(z-V)^{-1}$ such that $(z-V)^{-1} \in H(\Omega, \mathcal{L}_{\mathfrak{S}}(E))$ — with the usual conventions for the point at infinity. The *spectrum* $\sigma(V)$ of V is the complement of $\varrho(V)$ in the compactified complex plane $\mathbb{C}^*$:

$$\sigma(V) = \mathbb{C}^* \setminus \varrho(V).$$

The *spectral radius* $r(V)$ of V is defined as usual

$$r(V) = \sup_{z \in \sigma(V)} |z|.$$

Note that the domain of definition of $(z-V)^{-1}$ is strictly larger than $\varrho(V)$ in general, unlike in the case of Banach spaces. With this definition of $\sigma(V)$, however, the analogue of Gelfand's spectral radius formula holds:

Theorem 1 (Williamson, Waelbroeck, Neubauer)

$$r(V) = \sup_p \lim_{n \to \infty} \left(p(V^n)\right)^{1/n},$$

where p runs over a fundamental system of semi-norms generating the topology of $\mathcal{L}_{\mathfrak{S}}(E)$. $r(V)$ may take the value ∞ here.

Proof See [27].

We are now able to state the Riesz decomposition theorem for operators on Φ:

Theorem 2 *Let $r(V) < \infty$. If $\sigma(V) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint non-empty closed sets, then there are two simple closed positively oriented rectifiable Jordan curves $\mathcal{C}_1$ and $\mathcal{C}_2$, which lie in $\varrho(V)$ and contain in their interior σ_1 and σ_2, respectively, such that*

$$P_{\sigma_k} := \frac{1}{2\pi i} \oint_{\mathcal{C}_k} (z-V)^{-1}\, dz \qquad k = 1, 2,$$

belong to $\mathcal{L}(\Phi)$ and satisfy the following relations:

$$P_{\sigma_k}^2 = P_{\sigma_k} \qquad P_{\sigma_1} P_{\sigma_2} = P_{\sigma_2} P_{\sigma_1} \qquad P_{\sigma_1} + P_{\sigma_2} = I.$$

Moreover, $P_{\sigma_1}(\Phi)$ and $P_{\sigma_2}(\Phi)$ are topologically complementary closed subspaces invariant under V.

Proof See [16, IV.21–23].

In order to characterise the class of operators of interest we give the following definitions starting from the more familiar Banach space setting (see [17]).

Definition 1 A continuous operator V on a Banach space Φ is a *Fredholm operator* if the following holds:

1) V is a topological homomorphism;

2) the range $V(\Phi)$ is closed;

3) the dimension of the kernel of V is finite;

4) the co-dimension of the range of V is finite, i.e. dim $\Phi/_{V(\Phi)} < \infty$.

This definition originated from the attempts to find suitable generalisations of Fredholm's theory of integral equations [11]. For a compact integral operator V, for example, $I - K$ is a Fredholm operator and the equation

$$(I - K)x = y$$

is normally solvable. Ironically, this by now standard definition of a Fredholm operator implies that a Fredholm *integral* operator is just not a Fredholm operator. This motivated the following

Definition 2 A continuous operator V on a Banach space Φ is a *Riesz* operator if

1) for any non-zero complex number z

   i) $z - V$ is a Fredholm operator;

   ii) the ascent and descent of $z - V$ are finite, i.e. the chains $(z - V)^{-n}(0)$ and $(z - V)^n(\Phi)$ become stationary;

2) the eigenvalues of V form a finite set or a sequence converging to zero.

We see that the class of Riesz operators comprises those operators for which the Riesz-Schauder theory of compact operators is valid [32, 38]. In Banach spaces Riesz operators may be characterised by the following theorem of Ruston [36]

Theorem 3 *A continuous operator V on a Banach space Φ is a Riesz operator if and only if*

1) the operator valued function $z \mapsto (I - zV)^{-1}$ is meromorphic in the entire plane;

2) the residues at the poles are finite rank projectors.

See [36] for the *proof.*

Remark Ruston's result may be restated in terms of the resolvent of V. The operator V is a Riesz operator if, and only if, the resolvent $(z - V)^{-1}$ is meromorphic on $\mathbb{C} \setminus \{0\}$ with the Riesz projectors associated with the poles (see theorem 2) having finite dimensional ranges.

Ruston also showed that for Riesz operators a determinant theory analogous to Fredholm's determinant theory for integral equations [11] is valid.

The definitions given above can be generalised to locally convex topological vector spaces Φ. A continuous operator operator V on Φ is a *Fredholm operator* if it has properties 1–4 of definition 1. Similarly, the continuous operator V on Φ is a *Riesz operator* if 1–2 of definition 2 hold. Our class of Riesz operators on locally convex topological vector spaces coincides with Pietsch's *R-endomorphisms* [30] and de Bruyn's *Riesz transformations* [9]. Unfortunately, de Bruyn in an earlier paper [8] introduced a class of operators on topological vector spaces, which he called *'Riesz operators'* and which is different from our class of Riesz operators. Nevertheless, the various definitions coincide in the Banach space setting (see [17]). A further element of confusion in the locally convex scenario arises from the fact that Ruston's characterisation of Riesz operators is no longer valid here. We shall, therefore, define yet another class of operators:

Definition A continuous operator V on a locally convex topological vector space is a *Fredholm-Riesz operator* if 1 and 2 of theorem 3 hold.

Wrobel [43] showed that if Φ is quasi-complete and barrelled then the class of Fredholm-Riesz operators coincides with Pietsch's *F-endomorphisms* [30]. Thus, for Fredholm-Riesz operators on rigged Hilbert spaces both, the Riesz-Schauder theory and a determinant theory similar to Fredholm's determinant theory are valid. The class of Fredholm-Riesz operators, however, is in general strictly contained within the class of Riesz operators (see [43] for a discussion). Finally, we note that a continuous operator on a quasi-complete and barrelled locally convex space is Fredholm-Riesz if and only if its dual is Fredholm-Riesz with respect to the strong dual topology. Thus our concept of Fredholm-Riesz operators is well suited to the rigged Hilbert space formulation of resonances.

3 The Rényi map

The Rényi map being the paradigm of a 'chaotic' dynamical system has received considerable attention over the past thirty years. Rényi [31] was the first to investigate this map, or rather this family of maps, which are now named after him. After he had shown that they are ergodic with respect to Lebesgue measure, Rokhlin [34] proved that they are even exact. More recently these maps were re-investigated using the associated Frobenius-Perron operator of these maps. Roepstorff [33] realised that the Bernoulli polynomials are eigenfunctions of its Frobenius-Perron operator. These polynomial eigenfunctions were also obtained by Gaspard [13] through Euler's summation formula. Hasegawa and Saphir [18, 19], and Antoniou and Tasaki [3] derived a generalised spectral decomposition of the Frobenius-Perron operator using an algorithm [4] based on the spectral decomposition technique for large Poincaré non-integrable dynamical systems developed by the Brussels-Austin groups [29, 28].

The β-adic Rényi map S_β on the closed unit interval $[0,1]$ is the multiplication modulo 1 by the integer β:

$$S_\beta : [0,1] \to [0,1]$$

$$S_\beta x = \beta x \bmod 1 \text{ with } \beta \in \mathbb{N}, \beta > 1.$$

The map can also be considered as a dynamical system (S_β, λ), where λ denotes the Lebesgue measure on the unit interval. The evolution of a probability density $f \in L^1(\lambda)$ under the action of S_β is given by the Frobenius-Perron operator U_β of the map (see, e.g. [24]):

$$U_\beta : L^1(\lambda) \to L^1(\lambda)$$

$$U_\beta f = \frac{d}{d\lambda} \int_{S_\beta^{-1}(.)} f \, d\lambda,$$

where $\frac{d}{d\lambda}$ is the Radon-Nikodým derivative with respect to λ. In our case, U_β is given by

$$(U_\beta f)(x) = \beta^{-1} \sum_{i=0}^{\beta-1} f(\phi_i(x)). \tag{1}$$

Here, ϕ_i is the inverse of the Rényi map on its i-th interval of monotonicity, i.e.

$$\phi_i : [0,1] \to [0,1]$$

$$\phi_i(x) = \frac{x+i}{\beta},$$

for $0 \le i \le \beta - 1$. Note that the Frobenius-Perron operator can be defined on $L^p(\lambda)$ for $p \ge 1$ as well, and thus, in particular, on the Hilbert space $L^2(\lambda)$. Its adjoint $U_\beta^\dagger$ is just the Koopman operator of S_β (see [24]).

We consider the following test-function spaces:

- the strict inductive limit $\mathcal{P}$ of polynomials;
- the Banach space $\mathcal{E}_c$ $(c > 0)$ of entire functions of exponential type c;
- the inductive limit $\tilde{\mathcal{E}}_c$ $(c > 0)$ of entire functions of exponential type less than c;
- the Fréchet space $\mathcal{H}(D_r)$ $(r > 1)$ of functions analytic in the open disk with radius r;
- the Fréchet space $\mathcal{C}^\infty$ of infinitely differentiable functions on the closed unit interval.

Each of these spaces is densely and continuously embedded into the following:

$$\mathcal{P} \hookrightarrow \mathcal{E}_c \hookrightarrow \tilde{\mathcal{E}}_c \hookrightarrow \mathcal{E}_{c'} \hookrightarrow \tilde{\mathcal{E}}_{c'} \hookrightarrow \mathcal{H}(D_{r'}) \hookrightarrow \mathcal{H}(D_r) \hookrightarrow \mathcal{C}^\infty \hookrightarrow L^2$$

for $c < c'$,$r < r'$ and, hence, each of them is densely and continuously embedded into L^2. Furthermore, each of these spaces is complete and barrelled, and invariant under the Frobenius-Perron operator. In particular, we have the following

Proposition 1 *The Frobenius-Perron operator of the Rényi map is a Fredholm-Riesz operator on each of the spaces listed above. The eigenvalues of the restriction of the Frobenius-Perron to any of these spaces are simple, of geometric multiplicity one, and are given by* $(\frac{1}{\beta})^n, n \in \mathbb{N}_0$, *i.e. the spectra of the restrictions are the same.*

For the proof and a more detailed description of these spaces we refer to [5] and [6] (see also [39] for a discussion of the $\tilde{\mathcal{E}}$-spaces.)

We saw in the last section that isolated eigenvalues of finite algebraic multiplicity are poles of the resolvent with finite-rank residue. Thus, an analytic continuation of the resolvent of the Frobenius-Perron operator of the Rényi map restricted to the spaces listed above will yield all the Ruelle resonances of the system. The residues are rank-1 projectors, which may be obtained via contour integration. In particular, the analytic continuation procedure will give the same answers as the spectral decomposition technique.

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