A note on mixing in interval maps

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We construct a class of simple dynamical systems for which all correlation properties, i.e., the entire spectrum of the Perron-Frobenius operator, is accessible by analytical means. As an application we discuss an example of an interval map with a small number of branches where the decay of correlations can be made arbitrarily fast. The analysis sheds light on the problem of relating correlation decay with other dynamical quantities such as Lyapunov spectra.

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Introduction – The decay of correlations in dynamical systems, or to put it in experimentalists' terms, the lineshape in spectra, is one of the most fundamental topics in nonequilibrium statistical physics. The study of mechanisms which cause correlation decay and relaxation processes, i.e., the emergence of dissipative behaviour, has a long tradition in theoretical physics with significant impact for applications. Historically, the question of the emergence of irreversibility and the computation of relaxation rates can be attributed to the celebrated idea of fluctuation dissipation relations as first emphasised by Einstein [1] and then extended by Onsager to the regime close to equilibrium [2]. There have been numerous, mostly not entirely successful attempts to extend these concepts beyond the linear regime, aiming to establish how the underlying microscopic Hamiltonian structure impacts at the macroscopic irreversible level. Recent approaches focus on fluctuation relations (see [3] for a recent overview), but compelling experimental support for such concepts is still missing.

Understanding the occurrence of irreversibility in macroscopic systems based on first principles, i.e. starting from a microscopic level, remains a largely open problem. Simple mathematical toy models play an important role, to uncover the underlying formal structures which cause irreversibility [4]. Even abstract spectral theory is able to contribute to some of these issues. Such ideas are at the heart of the link between statistical physics and modern approaches in dynamical systems theory [5, 6], where both areas mutually benefit.

The aim of this article is to make a small contribution in this direction, by studying the relaxation rates in the most basic dynamical models from a rigorous perspective , and illustrating the occurring challenges. For that purpose we will construct a model with arbitrarily fast decay rates, keeping other relevant dynamical quantities almost unaffected.

Low dimensional maps facilitate studying the impact of macroscopic observations on correlation decay, and the link between decay rates and other dynamical quantities in some detail. Prominent examples in this context are models for anomalous dynamics, which are often based on the impact of marginally stable sets on the resulting algebraic correlation decay [7]. Here we are concerned with a simpler set-up of mixing dynamics, where correlation decay is always exponential and the corresponding transfer operators possess a spectral gap. It is impossible to give a fair account of the entire body of the mathematical literature but a substantial part of the area probably started with [8]. The point we want to stress is that it is surprisingly difficult to estimate the actual exponential decay rate from the properties of the dynamical system, e.g., from the shape of a one dimensional map.

It seems to be a common perception that the time scale of correlation decay in chaotic dynamical systems is related to sensitivity of the underlying dynamics, i.e., with Lyapunov exponents. Even though such relations are not as straightforward as suggested by the analysis of piecewise linear models [9], correlation decay times are often viewed as estimates for Lyapunov exponents [10]. One should, of course, keep in mind that on formal mathematical grounds no relation has to be expected as both time scales, the decorrelation time and the Lyapunov time, probe different formal properties of the underlying transfer operator, namely the spectral gap vs. the dependence of the largest eigenvalue on parameters. Thus, the cause of correlation decay and the underlying mechanisms which determine the actual time scales are quite blurred.

It is in fact difficult to estimate numerical values for relaxation rates from the shape of the map, and that task is not facilitated by the absence of any non-trivial example which can be solved by analytical means. Hence we will develop a new paradigmatic model class which can give new analytical insight in complex behaviour, in particular, with regards to relaxation properties. In this context we will also address the question whether one can achieve an arbitrarily fast correlation decay in interval maps.

A rigorous account – We consider analytic expanding full branch interval maps $T : [-1, 1] \to [-1, 1]$, i.e., maps which consist of analytic branches with slope larger than one such that each branch maps onto the entire interval $I = [-1, 1]$ (see, e.g., Fig. 1). Furthermore, we require that the map can be extended to an analytic map on the complex unit circle $\tau \colon S^1 \to S^1$, by $\tau(\exp(i\pi x)) = \exp(i\pi T(x))$; that is, the different branches match up in an analytic way. Maps of this type are known to have nice dynamical properties. In particular, every such map possesses an invariant measure μ which is given in terms of an analytic density, and the correlation function for analytic observables φ and ψ

$$C_{\varphi,\psi}(n) = \int (\varphi \circ T^n)\psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu, \qquad (1)$$

decays exponentially. The rate of decay is determined by the spectral properties of the Perron-Frobenius operator associated to T. If $\Phi_1, \ldots, \Phi_K$ are the inverse branches of the map T, we can choose a neighbourhood $D \subset \mathbb{C}$ of I such that the associated Perron-Frobenius operator

$$\mathcal{L}_I f = \sum_{k=1}^{K} \Phi_k' \cdot (f \circ \Phi_k) \qquad (2)$$

is well defined and compact when considered on the Banach space of bounded holomorphic functions on D equipped with the supremum norm [11].

It is well established but probably not widely known that the spectral properties of the Perron-Frobenius operator $\mathcal{L}_I$ are linked with the properties of the complex analytic map τ on the complex unit circle (see, e.g., the remark in [12]). Eigenvalues and eigenfunctions of $\mathcal{L}_I$ fall into two classes [13]. Eigenfunctions which are periodic correspond to eigenfunctions of the Perron-Frobenuis operator $\mathcal{L}_{S^1}$ of the associated map τ on the complex unit circle, and vice versa. In addition, there are eigenvalues which are given by the inverse powers of the slope of T at the fixed point at the interval endpoint, $(T'(-1))^{-n}$, $n \geq 1$. The corresponding eigenfunctions cannot be extended analytically to the complex unit circle. Hence, solving the eigenvalue problem for the operator $\mathcal{L}_I$ reduces to solving the corresponding problem of $\mathcal{L}_{S^1}$ and using the following relation between their spectra

$$\sigma(\mathcal{L}_I) = \sigma(\mathcal{L}_{S^1}) \cup \{(T'(-1))^{-n} : n \geq 1\}. \qquad (3)$$

The simplest example illustrating this is the well-known Bernoulli map on the interval $T(x) = 2x - \mathrm{sgn}(x)$, whose Perron-Frobenius operator has eigenvalues $1/2^n$, $n \geq 0$, which are caused by the fixed point slope. On the complex unit circle this maps translates to $\tau(z) = z^2$. This dynamical system only has the trivial spectrum $\{0, 1\}$, i.e., correlations for periodic analytic observables decay super-exponentially.

This dichotomy naturally raises the question whether there can be a relation between mixing rates and other measures of chaoticity such as entropy or Lyapunov exponents. Partial answers can be found, e.g., as upper bounds of the mixing rate in terms of the entropy for real analytic suspension flows [14], and in terms of Lyapunov exponents for piecewise linear Markov maps [15]. One purpose of this note is to show that there are expanding interval maps for which the exponential rate of decay can be made arbitrarily large whereas the Lyapunov exponent or the entropy remains bounded.

To address such an issue one needs a sufficiently rich class of models where the entire spectrum of the Perron-Frobenius operator is accessible. It is well known that analytic maps acting on the complex unit circle which are also analytic in the entire unit disk can be written as so-called finite Blaschke products [16]

$$\beta(z) = \exp(i\gamma) \prod_{k=1}^{K} \frac{z - a_k}{1 - z\bar{a}_k}, \quad (|a_k| < 1, K > 1). \qquad (4)$$

These induce full branch interval maps if the complex phase is chosen such that $\beta(-1) = -1$, as each factor has modulus one on the unit circle. Expansivity of the map is equivalent to β having a unique fixed point $z_* = \beta(z_*)$ within the complex unit disk. Such a fixed point then determines the invariant density of the interval map via the analytic expression

$$\rho(x) = (1 - |z_*|^2)/(2|\exp(i\pi x) - z_*|^2). \qquad (5)$$

Apart from these classical results it is a recent discovery [17] that this fixed point also determines the entire spectrum of the Perron-Frobenius operator associated to the corresponding map on the unit circle:

$$\sigma(\mathcal{L}_{S^1}) = \{1, 0\} \cup \{(\beta'(z_*))^n, \left(\overline{\beta'(z_*)}\right)^n : n \geq 1\}. \qquad (6)$$

Using eq.(3) all eigenvalues of the Perron-Frobenius operator on the interval, $\mathcal{L}_I$, are then easily computed by supplementing eq.(6) with the powers of the inverse slope at the interval endpoint. Finite Blaschke products thus constitute an ideal laboratory for testing certain aspects of dynamical systems theory as all relevant quantities are accessible.

Decay of autocorrelations – We shall now construct a dynamical system with arbitrarily fast correlation decay, but with the counter-intuitive constraint of leaving other dynamical quantities such as the Lyapunov exponent and the invariant density essentially unaffected. Following the previous section, the idea is to construct a Bernoulli-type map but with large fixed point slope. To this end we consider the finite Blaschke product

$$\beta(z) = z^2 \frac{z - b}{1 - bz} \qquad (7)$$

for $b \in (-1, 1)$. It is straightforward to compute that eq.(7) induces a three-to-one analytic expanding map

$$B(x) = 3x + \frac{2}{\pi} \arctan\left(\frac{b \sin(\pi x)}{1 - b\cos(\pi x)}\right) - 2m \qquad (8)$$

on the interval $I = [-1, 1]$. Here m labels the three branches of the map, where $m = -1$ if $x < -\arccos((1+$

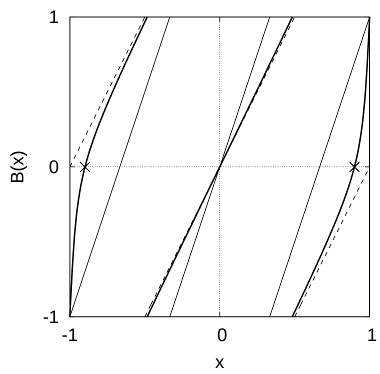

FIG. 1: The interval map in eq.(8) for parameter values $b = 0$ (thin solid) and $b = -0.95$ (thick solid). The dashed line shows the pointwise limit as $b \to -1$. The two crosses indicate the non-trivial zeros of the map for $b = -0.95$.

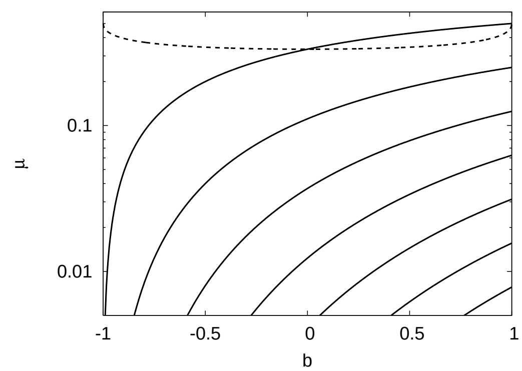

FIG. 2: Eigenvalues of the Perron-Frobenius operator in eq.(10) for the interval map in eq.(8), as a function of the parameter b (solid lines), in logarithmic scale. The dashed line shows the inverse Lyapunov multiplier $\exp(-\Lambda)$, see eq.(9).

$b)/2)/\pi$, $m = 1$ if $x > \arccos((1+b)/2)/\pi$, and $m = 0$ otherwise (see Fig. 1). The map B has fixed points at -1 and 1 with slope $B'(-1) = B'(1) = (3+b)/(1+b)$. For the trivial case $b = 0$, eq.(8) yields the tripling map, while a non-zero parameter value induces curvature. In the pointwise limit $b \to -1$, eq.(8) approaches a version of the Bernoulli map with offset. This limiting case will play an important role in our investigation.

The fixed point of eq.(7) within the unit disk is given by $z_* = 0$ for any value of the parameter b. Thus, the invariant density eq.(5) of the interval map eq.(8) is constant. Hence, for any $b \in (-1, 1)$ the Lyapunov exponent for B with respect to this invariant measure is given by $\Lambda = \int_I \ln B'(x)dx/2$ and with little effort can be calculated explicitly as

$$\Lambda = \ln\left(\left(3 + b^2 + \sqrt{(1-b^2)(9-b^2)}\right)/2\right), \qquad (9)$$

see Fig. 2. In addition, we have $\beta'(z_*) = 0$, and by eq.(6) there is no contribution from periodic eigenfunctions to the point spectrum of the Perron-Frobenius operator $\mathcal{L}_I$ in eq.(2). Thus all eigenvalues are caused by the inverse of the fixed point slope at $x = -1$ and we end up with

$$\sigma(\mathcal{L}_I) = \{1, 0\} \cup \{((1+b)/(3+b))^n : n \geq 1\} \; . \qquad (10)$$

The exponential rate of decay α for the correlation function in eq.(1) for generic analytic observables φ and ψ is related to the subleading eigenvalue $\mu_2 = (1+b)/(3+b)$ of $\mathcal{L}_I$ via $\alpha = -\ln \mu_2 = -\ln((1+b)/(3+b))$. Obviously, all the eigenvalues can be made arbitrarily small and the correlation decay arbitrarily fast, if the parameter b is chosen close to -1, while the Lyapunov exponent Λ in eq.(9) and the invariant density remain largely unaffected (see Fig. 2). Hence, α is not bounded by any finite multiple of Λ as b approaches -1, unlike the case of piecewise linear Markov maps considered in [15], where a bound $\alpha \leq 2\Lambda$ has been established.

How are the spectral properties discussed so far reflected in the actual shape of the correlation function? On the one hand, as all eigenvalues approach zero in the limit $b \to -1$ one expects a very fast asymptotic decay. On the other hand, the actual map B largely looks like a shifted version of the Bernoulli map (see Fig. 1) and one would naively expect an exponential decay according to the inverse slope. To clarify the picture let us look in some detail at the coordinate autocorrelation function, i.e., at eq.(1) for the observables $\phi(x) = \psi(x) = x$. As the mean value vanishes we are concerned with evaluating the integrals

$$C_{x,x}(n) = \frac{1}{2}\int_I x B^{(n)}(x)\, dx \, . \qquad (11)$$

It is fairly straightforward but slightly tedious to work out these integrals numerically to high precision. Given n, one computes cylinder sets of generation n and then performs the integral over each of these intervals with a suitable integration routine. For that purpose we have used a quadruple precision version of the QUADPACK routines [18]; the result is displayed in Fig. 3. The asymptotic decay of the correlation function is determined by the subleading eigenvalue. However, as the parameter b approaches -1 the autocorrelation function develops a pronounced transient exponential shape which follows the correlation decay of the shifted Bernoulli map. In this way the correlation function bridges the dichotomy pointed out in the previous paragraph.

There is a crossover between the transient (slow) exponential decay governed by the average slope of the map and the (fast) exponential decay determined by the maximal slope, i.e., by the subleading eigenvalue of the Perron-Frobenius operator. One can develop a simple heuristic argument to estimate the time scale at which this transition between the two different exponential regimes takes place. For the correlation function eq.(11) to be affected by the fine structure of the map

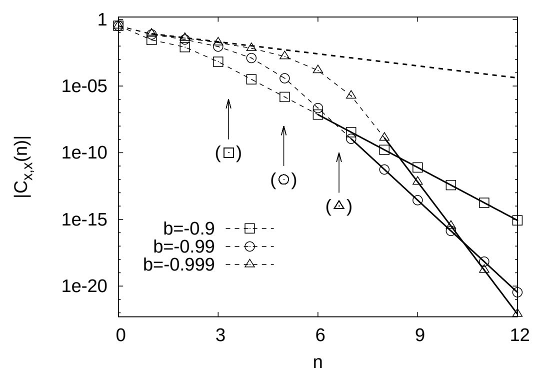

FIG. 3: Autocorrelation function eq.(11) of the map B in eq.(8) for different values of the parameter b (symbols with dashed lines as guide for the eye), on a logarithmic scale. Solid straight lines indicate the asymptotic decay as computed from the subleading eigenvalue of the Perron-Frobenius operator. The dashed straight line displays the correlation function $C_{x,x}(n) = -1/(6 \times 2^n)$ of the pointwise limit map as $b \to -1$ (see Fig. 1). Arrows show the respective estimates of the crossover time scale n_c according to eq.(12).

B, the time n has to be sufficiently large, allowing the dynamics to explore the small regions near $x = \pm 1$ where the map has large slope. The size Δ of these two intervals can be estimated by the zeros of the map close to $x = \pm 1$ (see Fig. 1). These zeros follow either from eq.(7) or eq.(8), yielding $\pm \arccos((b-1)/2)/\pi$ so that $\Delta = 1 - \arccos((b-1)/2)/\pi \simeq \sqrt{1+b}/\pi$ as $b \to -1$. From the dynamical perspective the correlation integral (11) at time n resolves phase space features which are averaged over cylinder sets of the corresponding generation. As in the limit $b \to -1$ the map has effectively slope 2, the size of these cylinder sets is given by 2^{-n}. Hence, the regions with large slope, where the dynamics differs from the shifted Bernoulli map, become relevant for the correlation function when $2^{-n} \simeq \sqrt{1+b}/\pi$, and we obtain for the crossover time scale the estimate

$$n_c \simeq -\ln(1+b)/(2\ln 2), \quad (b \to -1)\,. \qquad (12)$$

This simple argument predicts rather well the actual structure visible in Fig. 3. The formal challenge of course remains to develop a consistent and general mathematical framework for evaluating such features based on spectral properties of the Perron-Frobenius operator.

Conclusion – Given a dynamical system, the decay rate of correlations is difficult to predict, using simple features of the system only. The absence of analytically solvable examples which can be investigated in detail has hindered further progress in this direction in the past. The family of examples presented here constitutes a new paradigm and makes progress in this direction possible. It can be easily extended to account for more sophisticated features such as coupled systems and spatial degrees of freedom, thus shedding light on correlation decay in coupled complex structures.

Our example shows that even in the simplest possible setup, correlation decay can be made arbitrarily fast, independently of the Lyapunov exponent. Such an observation is somewhat counter-intuitive and shows that one cannot expect correlation decay to be caused by a simple mechanism. In particular, a strongly unstable fixed point on its own is by no means a sufficient condition for the fast decay reported here. As the analysis proves, a delicate global balance of dynamical features, which, in the present case, is induced by the required analytical properties, is essential. Nevertheless, expansivity, but not the Lyapunov exponent, is still one of the key ingredients for the decay of correlations. It is easy to confirm that for interval maps which can be extended analytically to the circle, the maximal expansion rate

$$\Lambda_+ = \lim_{n\to\infty} \sup \left\{ \frac{1}{n} \ln |(T^n)'(x)| : x \in I \right\} \qquad (13)$$

determines a lower bound for the subleading eigenvalue via $|\mu_2| \geq \exp(-\Lambda_+)$, or equivalently an upper bound for the mixing rate, $\alpha \leq \Lambda_+$. This rigorous statement is in line with previously reported relations between mixing rates and generalised Lyapunov spectra [9].

Although the spectrum of the Perron-Frobenius operator determines the asymptotic correlation decay, the present explicit example demonstrates that one has to be careful about its actual implications. On an intermediate time scale, probably relevant in real applications, the correlation decay may be governed by different mechanisms. Rigorous mathematical tools to cover such features and the observed crossover still need to be developed; pseudospectra could provide a suitable tool. A benchmark for success would be to capture all aspects of such a dynamically generated finite time scale phenomenon with the intrinsically generated dynamical crossover towards the asymptotic behaviour.

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