QUANTITATIVE SPECTRAL PERTURBATION THEORY FOR COMPACT OPERATORS ON A HILBERT SPACE

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Abstract. We introduce compactness classes of Hilbert space operators by grouping together all operators for which the associated singular values decay at a certain speed and establish upper bounds for the norm of the resolvent of operators belonging to a particular compactness class. As a consequence we obtain explicitly computable upper bounds for the Hausdorff distance of the spectra of two operators belonging to the same compactness class in terms of the distance of the two operators in operator norm.

1. Introduction

Perturbation theory is the study of the behaviour of characteristic data of a mathematical object when replacing it by a similar nearby object. More narrowly, spectral perturbation theory is concerned with the change of spectral data of linear operators (such as their spectrum, their eigenvalues and corresponding eigenvectors) when the operators are subjected to a small perturbation.

There are two sides to spectral perturbation theory, a qualitative one and a quantitative one. Qualitative perturbation theory focusses on questions such as the continuity, differentiability and analyticity of eigenvalues and eigenvectors, while quantitative perturbation theory attempts to provide computationally accessible bounds for the smallness of the change in the spectral data in terms of the smallness of the perturbation.

The book by Kato [Kat76] is the main reference for spectral perturbation theory, focussing mostly on the qualitative part of the theory. Qualitative and quantitative aspects are discussed in the article and book by Chatelin [Cha81, Cha83] and the book by Hinrichsen and Pritchard [HP11].

The present article, located at the interface of functional analysis and linear algebra, addresses the following problem of fundamental importance in both qualitative and quantitative perturbation theory. If A and B are two compact operators acting on a separable Hilbert space which are close, then how close are their spectra $\sigma(A)$ and $\sigma(B)$?

In order to make this question more precise we need to specify metrics to measure distances of operators and spectra. Distances of operators will typically be given by the underlying operator norm $\|\cdot\|$, while distances of spectra will be determined by the Hausdorff metric (see below).

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A standard result in qualitative perturbation theory tells us that if A and B are compact operators and $\|A-B\|$ becomes vanishingly small, then so does the Hausdorff distance of their spectra (see, for example, [New51, Theorem 3]). However, this result does not give any quantitative information on how large the Hausdorff distance of $\sigma(A)$ and $\sigma(B)$ is when $\|A-B\|$ is small but non-zero.

Quantitative information of this type is interesting in situations where one wants to determine the spectrum of an arbitrary compact operator A on a separable Hilbert space by numerical means. The standard approach to solving this infinite-dimensional problem is to reduce it to a finite-dimensional one. This can, for example, be achieved as follows. Fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for the Hilbert space and define orthogonal projections onto the space spanned by the first k basis vectors by setting

$$P_k x = \sum_{n=1}^{k} (x, e_n) e_n \,,$$

where $(\cdot,\cdot)$ denotes the inner product of H. Now

$$A_k = P_k A P_k$$

is a finite rank operator, the spectrum of which is in principle computable, at least to arbitrary precision, since it boils down to the computation of the eigenvalues of a matrix. Moreover, it is possible to show that this sequence of finite rank operators $(A_k)_{k\in\mathbb{N}}$ converges to A in operator norm (see, for example, [ALL01, Theorem 4.1]). Thus, if quantitative bounds for the Hausdorff distance of the spectra of two compact operators are available, then the spectrum of A can, in principle, be computed to arbitrary precision. In passing we note that the problem of determining the spectrum of an arbitrary bounded operator to a given precision is much more complicated (see [Han10]).

In order to formulate the results of this article we require some notation. For $z \in \mathbb{C}$ and a compact subset $\sigma \subset \mathbb{C}$ let

$$d(z,\sigma) = \inf_{\lambda\in\sigma} |z-\lambda|$$

denote the distance of z to σ. The *Hausdorff distance* $\mathrm{Hdist}(\cdot,\cdot)$, also known as the *Pompeiu-Hausdorff distance* (see [BP13] for some historical background), is the following metric defined on the set of compact subsets of $\mathbb{C}$

$$\mathrm{Hdist}(\sigma_1,\sigma_2) = \max\{\hat{d}(\sigma_1,\sigma_2), \hat{d}(\sigma_2,\sigma_1)\}$$

where

$$\hat{d}(\sigma_1,\sigma_2) = \sup_{\lambda\in\sigma_1} d(\lambda,\sigma_2)\,,$$

and σ_1 and σ_2 are two compact subsets of $\mathbb{C}$. It is easy to see that the Hausdorff distance is a metric on the set of compact subsets of $\mathbb{C}$.

Now recall the following notions from matrix perturbation theory: for two bounded operators A and B, the *spectral variation of A with respect to B* is defined to be

$$\hat{d}(\sigma(A),\sigma(B))\,,$$

while the *spectral distance of A and B* is

$$\mathrm{Hdist}(\sigma(A),\sigma(B))$$

(see, for example, [Gil03, Chapter 8, Definition 8.4.1]).

The main concern of the present article is to provide explicit upper bounds for the Hausdorff distance of the spectra of two arbitrary compact operators A and B on a separable Hilbert space in terms of the distance of the two operators A and B in operator norm. This will be achieved by grouping together all compact operators for which the associated singular values decay at a certain speed into a class, termed a *compactness class* (see Definition 3.2). These classes of operators generalise the exponential classes introduced by the second author (see [Ban08]).

Our approach relies on an adaptation of finite-dimensional arguments going back to work of Henrici (see [Hen62]), who obtained upper bounds for the spectral variation of two matrices as follows. In a first step, an upper bound for the norm of the resolvent $(zI - A)^{-1}$ of a matrix A is obtained which only depends on the distance of z to the spectrum of A. This is achieved by writing the matrix A as a perturbation of a normal matrix D having the same spectrum as A by a nilpotent matrix N. Explicit upper bounds for the spectral distance of two matrices can then be obtained in a second step, by using an argument going back to Bauer and Fike [BF60], which converts resolvent bounds into spectral distance bounds (see Theorem 6.1).

So far, infinite-dimensional analogues of these bounds have been obtained only for certain subclasses of compact operators. To the best of our knowledge, the first results in this direction are due to Gil', who, in a series of papers begun in 1979, obtained spectral variation and distance bounds mostly for operators in the Schatten classes (see [Gil95, Gil03] and references therein) and more recently for operators with inverses in the Schatten classes (see [Gil12, Gil14]). Pokrzywa [Pok85] has found similar bounds for operators in symmetrically normed ideals, while the second author obtained bounds, simpler and sharper than those of Gil' and Pokrzywa, for Schatten class operators [Ban04] and for operators in exponential classes [Ban08]. All three authors essentially use Henrici's approach to obtain their bounds, by first deriving resolvent bounds for quasi-nilpotent operators and then using the perturbation argument outlined above. For a completely different approach to obtain spectral variation bounds using determinants, see [BG15].

This article is organised as follows. In Section 3 we give the precise definition of compactness classes determined by the speed of decay of the singular values of the operators in the class and study their functional analytic properties in some detail. In particular we shall find sufficient conditions guaranteeing that these classes of operators form quasi-Banach operator ideals in the sense of Pietsch (see [Pie80, Pie86]). In Section 4 we shall use a theorem of Dostanić [Dos01] to produce bounds for the resolvents of quasi-nilpotent operators in a given compactness class. Using the technique of Henrici discussed earlier we then obtain an upper bound for $\left\|(zI - A)^{-1}\right\|$ for an arbitrary operator A in a given compactness class, which depends only on the asymptotics of the singular values of A and the distance of z to the spectrum of A (see Theorem 4.12). The following Section 5 is devoted to studying the behaviour of the bound for the norm of resolvents derived in the previous section for two particular families of compactness classes already in the literature. In Section 6, the general resolvent bounds obtained in Section 4 together with the Bauer-Fike argument will yield the main result of this article, an explicit upper bound for the spectral distance of two operators in a given compactness class, depending only on the distance in operator norm of the operators and their respective departures from normality (see Theorem 6.2). To the best of our knowledge, no bound for the spectral distance applicable to arbitrary compact operators has appeared in the literature yet. A particular feature of this result is that it turns out to be sharp for normal operators (see Remark 6.3 (iii)). In the final section we will briefly discuss an application of the main result giving circular inclusion regions for pseudospectra of an operator in a given compactness class (see Theorem 7.2).

2. Preliminaries

In this section we fix notation and briefly recapitulate some facts about compact operators on a Hilbert space which we rely on in the following.

Let H_1 and H_2 be separable Hilbert spaces. We write $L(H_1, H_2)$ to denote the Banach space of bounded linear operators from H_1 to H_2 equipped with the operator norm $\|\cdot\|$ and $S_\infty(H_1, H_2) \subset L(H_1, H_2)$ to denote the closed subspace of compact operators from H_1 to H_2. If $H = H_1 = H_2$ we use the short-hands $L(H)$ and $S_\infty(H)$ for $L(H_1, H_2)$ and $S_\infty(H_1, H_2)$, respectively.

For $A \in L(H)$ the spectrum and the resolvent set of A will be denoted by $\sigma(A)$ and $\rho(A)$, respectively. Moreover, for $z \in \rho(A)$, we write $R(A; z) = (zI - A)^{-1}$ for the resolvent of A.

For $A \in S_\infty(H)$ we use $\lambda(A) = (\lambda_k(A))_{k\in\mathbb{N}}$ to denote its eigenvalue sequence, counting algebraic multiplicities and ordered by decreasing modulus so that

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots$$

If A has only finitely many non-zero eigenvalues, we set $\lambda_k(A) = 0$ for $k > N$, where N denotes the number of non-zero eigenvalues of A. The symbol $|\lambda(A)|$ will denote the sequence $(|\lambda_k(A)|)_{k\in\mathbb{N}}$.

Let now $A \in S_\infty(H_1, H_2)$. For $k \in \mathbb{N}$, the k-th singular value of A is given by

$$s_k(A) = \sqrt{\lambda_n(A^*A)} \quad (k \in \mathbb{N}),$$

where A^* denotes the Hilbert space adjoint of A. For later use, we note that the singular values enjoy the following two properties. Given $A \in S_\infty(H_3, H_2)$, $B \in L(H_1, H_2)$ and $C \in L(H_4, H_3)$ we have

$$s_k(BAC) \leq \|B\|\, s_k(A)\, \|C\| \qquad (\forall k \in \mathbb{N}), \tag{1}$$

while for $A, B \in S_\infty(H_1, H_2)$ we have

$$s_{k+l-1}(A + B) \leq s_k(A) + s_l(B) \quad (\forall k, l \in \mathbb{N}). \tag{2}$$

Eigenvalues and singular values satisfy a number of inequalities known as Weyl's inequalities. We give the most important one, known as the multiplicative Weyl inequality (see [GGK90, Chapter VI, Theorem 2.1]).

Let $A \in S_\infty(H)$. Then we have

$$\prod_{k=1}^{n} |\lambda_k(A)| \leq \prod_{k=1}^{n} s_k(A) \quad (\forall n \in \mathbb{N}). \tag{3}$$

For more information about these notions see, for example, [DS63, GK69, Pie86].

3. Compactness Classes

The basic idea to define these classes is to group together all compact operators on a separable Hilbert space the singular values of which decay at a certain speed, quantified by a given 'weight sequence' (see Definition 3.2).

The aim of this section is to examine the behaviour of compactness classes under addition and multiplication, to show that these classes are quasi-Banach operator ideals under suitable conditions on the weight sequence and to determine the decay rate of the eigenvalue sequence of an operator in a given compactness class.

We start by defining the notion of a weight sequence.

Definition 3.1. Let

$$\mathcal{W} = \{\, w : \mathbb{N} \to \mathbb{R}_0^+ \, : \, w_k \geq w_{k+1}, \, \forall k \in \mathbb{N} \text{ and } \lim_{k\to\infty} w_k = 0 \,\} \, .$$

Elements of $\mathcal{W}$ will be referred to as *weight sequences*, or simply *weights*.

Every $w \in \mathcal{W}$ now gives rise to a compactness class as follows.

Definition 3.2. Let $w \in \mathcal{W}$. An operator $A \in S_\infty(H_1, H_2)$ is said to be *w-compact* if there is a constant $M \geq 0$ such that

$$s_k(A) \leq M w_k \quad (\forall k \in \mathbb{N}) \, . \tag{4}$$

The infimum over all M such that (4) holds will be referred to as the *w-gauge of A* and will be denoted by $|A|_w$.

The collection of all w-compact operators $A \in S_\infty(H_1, H_2)$ will be denoted by $E_w(H_1, H_2)$ or simply by $E_w(H)$ in case $H = H_1 = H_2$.

For later use we also define the following sequence space analogues of compactness classes.

Definition 3.3. Given $w \in \mathcal{W}$, let $\mathcal{E}_w$ denote the set of all complex-valued sequences $(x_n)_{n\in\mathbb{N}}$ for which there is a constant $M \geq 0$ such that

$$|x_k| \leq M w_k \quad (\forall k \in \mathbb{N}) \, . \tag{5}$$

The infimum over all M such that (5) holds will be referred to as the *w-gauge of x* and will be denoted by $|x|_w$.

Remark 3.4. It is not difficult to see that $\mathcal{E}_w$ is a Banach space when equipped with the w-gauge $|\cdot|_w$. The situation is different for E_w, which need not even be a linear space in general (see Proposition 3.12).

Compactness classes generalise classes that have already appeared in the literature, such as the Schatten-Lorentz ideals $S_{p,\infty}$ (see, for example, [Pel85, p. 481]), which correspond to the weights $w_k = k^{-1/p}$ with $p \in (0, \infty)$ or the 'exponential classes' studied by Bandtlow (see [Ban08]), which correspond to weights of the form $w_k = \exp(-ak^\alpha)$ with $a \in (0, \infty)$ and $\alpha \in (0, \infty)$.

We shall now explore some of the properties of $E_w(H_1, H_2)$ for a general weight w. We start with the following elementary observation.

Proposition 3.5. *Let $v, w \in \mathcal{W}$. If there exists $M \geq 0$ such that $v_k \leq M w_k$ for every $k \in \mathbb{N}$ and $A \in E_v(H_1, H_2)$, then $A \in E_w(H_1, H_2)$ and $|A|_w \leq M |A|_v$.*

Proof. Suppose $A \in E_v(H_1, H_2)$ and there exists $M \geq 0$ such that $v_k \leq M w_k$ for every $k \in \mathbb{N}$. Then we have, for every $k \in \mathbb{N}$,

$$s_k(A) \leq |A|_v v_k \leq |A|_v M w_k \, .$$

Hence we obtain $A \in E_w(H_1, H_2)$ and $|A|_w \leq M |A|_v$. ☐

The observation above motivates defining a partial order on $\mathcal{W}$ as follows

$$v \preceq w :\iff \exists M \geq 0 \quad \text{such that} \quad v_k \leq M w_k \quad (\forall k \in \mathbb{N})\,.$$

We shall also define an equivalence relation on $\mathcal{W}$ by setting

$$v \asymp w :\iff v \preceq w \quad \text{and} \quad w \preceq v\,.$$

Using the above partial order we obtain the following inclusion.

Proposition 3.6. *Let* $\dim H_1 = \dim H_2 = \infty$ *and let* $v, w \in \mathcal{W}$. *Then*

$$v \preceq w \iff E_v(H_1, H_2) \subseteq E_w(H_1, H_2)\,.$$

Proof. For the forward implication we need to show that if $v \preceq w$ then $E_v(H_1, H_2) \subseteq E_w(H_1, H_2)$. This, however, follows directly from Proposition 3.5.

For the converse, suppose that $E_v(H_1, H_2) \subseteq E_w(H_1, H_2)$. We need to show that $v \preceq w$. Fix orthonormal bases $(e_k)_{k\in\mathbb{N}}$ for H_1 and $(f_k)_{k\in\mathbb{N}}$ for H_2. Define an operator $A \in L(H_1, H_2)$ by setting $Ae_k = v_k f_k$ for every $k \in \mathbb{N}$. We clearly have $s_k(A) = v_k$ for every $k \in \mathbb{N}$, so $A \in E_v(H_1, H_2)$. But since $E_v(H_1, H_2) \subseteq E_w(H_1, H_2)$, we have $A \in E_w(H_1, H_2)$. Thus there exists $M \geq 0$ such that $v_k = s_k(A) \leq M w_k$ for every $k \in \mathbb{N}$, so $v \preceq w$ and the backwards implication is proved as well. □

Corollary 3.7. *Let* $\dim H_1 = \dim H_2 = \infty$ *and let* $v, w \in \mathcal{W}$. *Then*

$$v \asymp w \iff E_v(H_1, H_2) = E_w(H_1, H_2)\,.$$

Although $E_w(H_1, H_2)$ is not a linear space in general, it is closed under multiplication by scalars and operators, as we shall see presently.

Lemma 3.8. *If* $A \in E_w(H_1, H_2)$ *and* $\alpha \in \mathbb{C}$, *then*

$$\alpha A \in E_w(H_1, H_2) \quad \textit{and} \quad |\alpha A|_w = |\alpha|\, |A|_w\,.$$

Proof. Let $A \in E_w(H_1, H_2)$ and $\alpha \in \mathbb{C}$. Then

$$s_k(\alpha A) = |\alpha| s_k(A) \leq |\alpha| |A|_w w_k \quad (\forall k \in \mathbb{N})\,,$$

so $\alpha A \in E_w(H_1, H_2)$ and

$$|\alpha A|_w \leq |\alpha| |A|_w\,. \tag{6}$$

It remains to prove that $|\alpha A|_w \geq |\alpha||A|_w$ for every $\alpha \in \mathbb{C}$. If $\alpha = 0$, then there is nothing to prove. If $\alpha \neq 0$, then using (6) we have

$$|A|_w = |\alpha^{-1}\alpha A|_w \leq |\alpha^{-1}||\alpha A|_w\,.$$

Therefore we obtain, for every $\alpha \in \mathbb{C}$,

$$|\alpha||A|_w \leq |\alpha A|_w\,.$$

□

Proposition 3.9. *If* $B \in L(H_2, H_1)$, $A \in E_w(H_3, H_2)$ *and* $C \in L(H_4, H_3)$, *then* $|BAC|_w \leq \|B\|\, |A|_w\, \|C\|$. *And*

$$L(H_2, H_1) E_w(H_3, H_2) L(H_4, H_3) \subseteq E_w(H_4, H_1)\,.$$

Proof. Let $A \in E_w(H_3, H_2)$. By (1), we obtain

$$s_k(BAC) \leq \|B\|\, s_k(A)\, \|C\| \leq \|B\|\, |A|_w\, \|C\|\, w_k \quad (\forall k \in \mathbb{N})\,.$$

Thus we have $BAC \in E_w(H_4, H_1)$ and $|BAC|_w \leq \|B\|\, |A|_w\, \|C\|$. □

Remark 3.10. Note that Proposition 3.9 implies that

$$L(H)E_w(H)L(H) \subseteq E_w(H)\,.$$

Hence $E_w(H)$ satisfies the second condition of the definition of an operator ideal (see, for example, [Pie80, 1.1.1]) though not necessarily the first one, concerned with linearity. Thus $E_w(H)$ is what is sometimes referred to as a pre-ideal (see, for example, [Nel82]).

We shall now investigate the behaviour of compactness classes under addition (see Proposition 3.12). Before doing so we require the following definition.

Definition 3.11. Let $w \in \mathcal{W}$. Then $\dot{w}$ is the sequence obtained from w by doubling each entry, that is, $\dot{w} = (w_1, w_1, w_2, w_2, w_3, w_3, \ldots)$. More precisely, $\dot{w}$ is the sequence given by

$$\dot{w}_k = \begin{cases} w_{\frac{k}{2}} & \text{if } k \text{ is even,} \\ w_{\frac{k+1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

We are now ready to investigate how compactness classes behave under addition.

Proposition 3.12. *Let $w \in \mathcal{W}$. Then the following assertions hold.*

(i) *If $A, B \in E_w(H_1, H_2)$, then $A + B \in E_{\dot{w}}(H_1, H_2)$ with*

$$|A + B|_{\dot{w}} \leq |A|_w + |B|_w\,.$$

(ii) *If $\dim H_1 = \dim H_2 = \infty$, then assertion (i) is sharp in the sense that if there is $v \in \mathcal{W}$ such that $A + B \in E_v(H_1, H_2)$ for all $A, B \in E_w(H_1, H_2)$, then $\dot{w} \preceq v$.*

Proof.

(i) Suppose $A, B \in E_w(H_1, H_2)$. Using (2) we have

$$s_{2k-1}(A+B) \leq s_k(A) + s_k(B) \leq (|A|_w + |B|_w)w_k = (|A|_w + |B|_w)\dot{w}_{2k-1}$$

since $\dot{w}_{2k-1} = w_k$ for every $k \in \mathbb{N}$. As the singular values are monotonically decreasing and $\dot{w}_{2k} = w_k$ for every $k \in \mathbb{N}$, we obtain

$$s_{2k}(A+B) \leq s_{2k-1}(A+B) \leq (|A|_w + |B|_w)w_k = (|A|_w + |B|_w)\dot{w}_{2k}\,.$$

Hence we have

$$s_k(A+B) \leq (|A|_w + |B|_w)\dot{w}_k \quad (\forall k \in \mathbb{N})\,.$$

Therefore

$$A + B \in E_{\dot{w}} \quad \text{and} \quad |A+B|_{\dot{w}} \leq |A|_w + |B|_w\,.$$

(ii) Since both H_1 and H_2 are infinite-dimensional we can choose orthonormal bases $(e_k)_{k\in\mathbb{N}}$ for H_1 and $(f_k)_{k\in\mathbb{N}}$ for H_2. Define an operator $A \in L(H_1, H_2)$ by setting

$$Ae_k = \begin{cases} 0 & \text{if } k \text{ is even,} \\ w_{\frac{k+1}{2}} f_k & \text{if } k \text{ is odd,} \end{cases}$$

and an operator $B \in L(H_1, H_2)$ by setting

$$Be_k = \begin{cases} w_{\frac{k}{2}} f_k & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Clearly, we have

$$s_k(A) = s_k(B) = w_k \quad (\forall k \in \mathbb{N}),$$

so $A, B \in S_\infty(H_1, H_2)$. At the same time we have

$$s_k(A+B) = \dot{w}_k \quad (k \in \mathbb{N}),$$

so $A + B \in E_v(H_1, H_2)$. Using the observation above, there exists $M \geq 0$ such that, for every $k \in \mathbb{N}$,

$$\dot{w}_k = s_k(A+B) \leq M v_k ,$$

which means $\dot{w} \preceq v$.

□

The proposition above implies that $E_w(H_1, H_2)$ is not a linear space in general. However, it points towards a simple sufficient condition guaranteeing linearity.

Corollary 3.13. *If $\dot{w} \asymp w$, then $E_w(H_1, H_2)$ is a linear space and $|\cdot|_w$ is a quasi-norm.*

Proof. By Lemma 3.8, we have $\alpha A \in E_w(H_1, H_2)$ for every $\alpha \in \mathbb{C}$ and $A \in E_w(H_1, H_2)$. Moreover, using Proposition 3.12 and the assumption $\dot{w} \asymp w$ we have

$$A + B \in E_{\dot{w}}(H_1, H_2) = E_w(H_1, H_2).$$

Thus $E_w(H_1, H_2)$ is a linear space. It remains to show that $|\cdot|_w$ is a quasi-norm. The only non-trivial property is the quasi-triangle inequality, that is, we need to show that there is $M > 0$ such that

$$|A + B|_w \leq M(|A|_w + |B|_w) \quad (\forall A, B \in E_w(H_1, H_2)).$$

In order to see this note that, since $\dot{w} \asymp w$ there exists $M \geq 1$ such that

$$\frac{1}{M}|A|_w \leq |A|_{\dot{w}} \leq M|A|_w$$

for every $A \in E_w(H_1, H_2) = E_{\dot{w}}(H_1, H_2)$. Since $A, B \in E_w(H_1, H_2)$ then, by Proposition 3.12, we have $|A + B|_{\dot{w}} \leq |A|_w + |B|_w$. It follows that if $A, B \in E_w(H_1, H_2)$, then $A + B \in E_{\dot{w}}(H_1, H_2) = E_w(H_1, H_2)$ and

$$\frac{1}{M}|A + B|_w \leq |A + B|_{\dot{w}} \leq |A|_w + |B|_w .$$

□

Proposition 3.14. *If $\dot{w} \asymp w$, then $E_w(H_1, H_2)$ is complete with respect to the quasi-norm $|\cdot|_w$.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $E_w(H_1, H_2)$. First we note that $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $S_\infty(H_1, H_2)$ with respect to the operator norm $\|\cdot\|$, since $\|A_n - A_m\| \leq |A_n - A_m|_w w_1$. As $S_\infty(H_1, H_2)$ is complete there is an $A \in S_\infty(H_1, H_2)$ such that $A_n \to A$ as $n \to \infty$ in the operator norm $\|\cdot\|$. We need to prove that $A \in E_w(H_1, H_2)$ and $|A_n - A|_w \to 0$ as $n \to \infty$. Fix $\epsilon \geq 0$. Since $(A_n)_{n\in\mathbb{N}}$ is Cauchy in $|\cdot|_w$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$s_k(A_n - A_m) \leq |A_n - A_m|_w w_k \leq \epsilon w_k \quad (\forall n, m \geq N_\epsilon, \forall k \in \mathbb{N}).$$

Letting $m \to \infty$ in the above we obtain

$$s_k(A_n - A) \leq \epsilon w_k \quad (\forall n \geq N_\epsilon, \forall k \in \mathbb{N}),$$

and so

$$|A_n - A|_w \leq \epsilon \quad (\forall n \geq N_\epsilon) . \tag{7}$$

The above implies that $|A_n - A|_w \to 0$ as $n \to \infty$. It remains to show that $A \in E_w(H_1, H_2)$. In order to see this, fix $n \geq N_\epsilon$. Inequality (7) now implies that $A_n - A$ is an element of $E_w(H_1, H_2)$. Since A_n is also an element of $E_w(H_1, H_2)$ and $E_w(H_1, H_2)$ is linear by Corollary 3.13, we then obtain $A \in E_w(H_1, H_2)$. □

Proposition 3.15. *If $\dot{w} \asymp w$, then E_w is a quasi-Banach operator ideal.*

Proof. Follows from Proposition 3.9, Corollary 3.13 and Proposition 3.14. □

We now turn to studying the rate of decay of the eigenvalue sequence of an operator in a given compactness class. In order to do this we require the following notation.

Definition 3.16. Let $w \in \mathcal{W}$. Then we define $\bar{w}$ as the sequence of successive geometric means of w, that is,

$$\bar{w}_k = (w_1 \cdots w_k)^{\frac{1}{k}} \quad (\forall k \in \mathbb{N}) .$$

Proposition 3.17. *Let $A \in E_w(H_1, H_2)$. Then*

$$\lambda(A) \in \mathcal{E}_{\bar{w}} \quad \textit{with} \quad |\lambda(A)|_{\bar{w}} \leq |A|_w .$$

Proof. Let $A \in E_w(H_1, H_2)$. By the multiplicative Weyl inequality (3) we have, for every $k \in \mathbb{N}$,

$$|\lambda_k(A)|^k \leq \prod_{l=1}^{k} |\lambda_l(A)| \leq \prod_{l=1}^{k} s_l(A) \leq |A|_w w_1 \cdots |A|_w w_k \leq |A|_w^k \, w_1 \cdots w_k .$$

Thus

$$|\lambda_k(A)| \leq |A|_w (w_1 \cdots w_k)^{\frac{1}{k}} = |A|_w \bar{w}_k \quad (\forall k \in \mathbb{N}) ,$$

and we obtain

$$\lambda(A) \in \mathcal{E}_{\bar{w}} \quad \text{and} \quad |\lambda(A)|_{\bar{w}} \leq |A|_w ,$$

as desired.

□

4. General resolvent bounds

The first bound for the norm of the resolvent of a linear operator on an infinite-dimensional Hilbert space was derived by Carleman (see [Car21]), who obtained a bound for Hilbert-Schmidt operators. His result was later generalised to Schatten-von Neumann operators (see, for example, [DS63, Sim77]). For more information about generalised Carleman type estimates see also [DP94, DP96].

In this section we shall derive an upper bound for the norm of the resolvent $R(A; z)$ of $A \in E_w(H_1, H_2)$ in terms of the distance of z to the spectrum of A and the w-departure from normality of A, a number measuring the non-normality of A. As already mentioned, we shall generalise the approach of Henrici in [Hen62] outlined in the introduction to the infinite-dimensional setting. The basic idea will be to write A as a sum of a normal operator D with $\sigma(D) = \sigma(A)$ and a quasi-nilpotent operator N, that is, an operator the spectrum of which consists of the point 0 only, and to consider A as a perturbation of D by N.

We start with a bound for powers of quasi-nilpotent operators, due to Dostanić.

Theorem 4.1. *There is a constant $C \geq \pi/2$ such that for any quasi-nilpotent $A \in S_\infty(H)$ and for every $k \in \mathbb{N}$ we have*

$$\left\|A^{2k}\right\| \leq C^{2k}(s_1(A)\cdots s_k(A))^2 .$$

Proof. See [Dos01, Theorem 1]. □

Given $w \in \mathcal{W}$, we define a function $F_w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by setting

$$F_w(r) = (1 + rw_1)\left(1 + \sum_{k=1}^{\infty}(w_1\cdots w_k)^2(Cr)^{2k}\right) , \tag{8}$$

where C is the constant from Theorem 4.1. It is not difficult to see that F_w is well-defined, real-analytic and strictly monotonically increasing. We are now ready to deduce resolvent bounds for quasi-nilpotent operators.

Proposition 4.2. *Let $w \in \mathcal{W}$ and let $A \in E_w(H)$ be a quasi-nilpotent operator. Then*

$$\left\|(I - A)^{-1}\right\| \leq F_w(|A|_w) .$$

Proof. Suppose $A \in E_w(H)$ is a quasi-nilpotent operator. Using a Neumann series and Theorem 4.1, we have

$$\begin{aligned}\left\|(I - A)^{-1}\right\| &\leq \sum_{k=0}^{\infty}\left\|A^k\right\| = \sum_{k=0}^{\infty}(\left\|A^{2k}\right\| + \left\|A^{2k+1}\right\|) ,\\ &\leq (1 + \|A\|)\left(1 + \sum_{k=1}^{\infty}\left\|A^{2k}\right\|\right) ,\\ &\leq (1 + s_1(A))\left(1 + \sum_{k=1}^{\infty}C^{2k}(s_1(A)\cdots s_k(A))^2\right) .\end{aligned}$$

Therefore we obtain

$$\left\|(I - A)^{-1}\right\| \leq (1 + |A|_w w_1)\left(1 + \sum_{k=1}^{\infty}(w_1\cdots w_k)^2(C|A|_w)^{2k}\right) ,$$

as required. □

An immediate consequence of the previous proposition is the following estimate for the growth of the resolvent of a quasi-nilpotent operator $A \in E_w(H)$.

Corollary 4.3. *Let $w \in \mathcal{W}$ and let $A \in E_w(H)$ be quasi-nilpotent. Then for any $z \neq 0$*

$$\|R(A;z)\| \leq |z|^{-1} F_w(|z|^{-1}|A|_w) .$$

By means of the following theorem, an upper bound for the norm of the resolvent $R(A;z)$ of $A \in E_w(H_1, H_2)$ can be obtained.

Theorem 4.4. *Let $A \in S_\infty(H)$. Then A can be written as a sum*

$$A = D + N ,$$

such that

(i) *$D \in S_\infty(H), N \in S_\infty(H)$;*
(ii) *D is normal and $\lambda(D) = \lambda(A)$;*
(iii) *N and $(zI - D)^{-1}N$ are quasi-nilpotent for every $z \in \rho(D) = \rho(A)$.*

Proof. See [Ban04, Theorem 3.2]. □

The theorem above motivates the following definition.

Definition 4.5. Let $A \in S_\infty(H)$. A decomposition

$$A = D + N$$

with D and N satisfying the properties (i–iii) of the previous theorem is called a *Schur decomposition of A*. We call the operators D and N the *normal* and the *quasi-nilpotent part of the Schur decomposition of A*, respectively.

Remark 4.6. The decomposition is not unique, as can be seen from the following example taken from [Ban04, Remark 3.5 (i)]. Consider

$$A := \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:D_1} + \underbrace{\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}_{=:N_1}$$
$$= \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:D_2} + \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}}_{=:N_2} .$$

It is easy to see that D_1 and D_2 are normal and that N_1 and N_2 are nilpotent. Moreover $\sigma(A) = \sigma(D_1) = \sigma(D_2) = \{2, 0\}$. Furthermore, both $(zI - D_1)^{-1}N_1$ and $(zI - D_2)^{-1}N_2$ are nilpotent for any $z \in \rho(A)$. Thus A has two different Schur decompositions.

Note that the normal parts are obviously unitarily equivalent. However, the nilpotent parts are not. In order to see this observe that

$$\|N_1\|_4^4 = 112 \neq 80 = \|N_2\|_4^4 ,$$

where $\|\cdot\|_4$ is the norm of the Schatten class $S_4(\mathbb{C}^3)$.

In the following proposition we determine an upper bound for the singular values of the normal part and the quasi-nilpotent part of a Schur decomposition of an operator in a given compactness class.

Proposition 4.7. *Let $A \in E_w(H)$. If $A = D + N$ is a Schur decomposition of A with normal part D and quasi-nilpotent part N, then*

(i) *$D \in E_{\bar{w}}(H)$ with $|D|_{\bar{w}} \leq |A|_w$, where $\bar{w}_k = (w_1 \cdots w_k)^{\frac{1}{k}}$.*
(ii) *$N \in E_{\dot{\bar{w}}}(H)$ with $|N|_{\dot{\bar{w}}} \leq 2|A|_w$, where $\dot{\bar{w}} = (w_1, w_1, (w_1w_2)^{\frac{1}{2}}, (w_1w_2)^{\frac{1}{2}}, \dots)$.*

Proof. Let $A \in E_w(H)$. Since D is normal, its singular values coincide with the moduli of its eigenvalues, which also coincide with the moduli of the eigenvalues of A. Using Proposition 3.17 and the fact that D is normal we obtain

$$s_k(D) \leq |A|_w \bar{w}_k ,$$

so $D \in E_{\bar{w}}(H)$ and $|D|_{\bar{w}} \leq |A|_w$, as required.

For the second part, observe that since $w \preceq \bar{w}$, we have $|A|_{\bar{w}} \leq |A|_w$ via Proposition 3.5. Then we also have $A \in E_{\bar{w}}(H)$ by Proposition 3.6. Thus, using Proposition 3.12, we have

$$N = A - D \in E_{\dot{\bar{w}}} ,$$

$$|A-D|_{\dot{\bar{w}}} \leq |A|_{\bar{w}} + |D|_{\bar{w}}$$

and so, using assertion (i), we obtain

$$|N|_{\dot{\bar{w}}} \leq |A|_w + |A|_w = 2|A|_w\,,$$

as desired. □

We now define the analogue of Henrici's departure from normality for operators in a given compactness class.

Definition 4.8. Let $w \in \mathcal{W}$ and $A \in E_w(H)$. Then

$$\nu_w(A) = \inf\{\,|N|_{\dot{\bar{w}}} \,:\, N \text{ is the quasi-nilpotent part of a Schur decomposition of } A\,\}$$

is called the *w-departure from normality of A*.

Remark 4.9. Note that by the previous proposition, the w-departure from normality of an operator in $E_w(H)$ is always finite.

The term 'departure from normality' is justified in view of the following proposition.

Proposition 4.10. *Let $A \in E_w(H)$. Then*

$$A \textit{ is normal} \iff \nu_w(A) = 0\,.$$

Proof. The forward implication is trivial. For the backwards implication, let $\nu_w(A) = 0$. Then there exists a sequence of Schur decompositions with quasi-nilpotent parts N_n such that $|N_n|_{\dot{\bar{w}}} \to 0$ as $n \to \infty$. But

$$\|A - D_n\| = \|N_n\| = s_1(N_n) \leq \dot{\bar{w}}_1 |N_n|_{\dot{\bar{w}}}\,,$$

where D_n are the corresponding normal parts, so $\lim_{n\to\infty} \|A - D_n\| = 0$. Hence A is a limit of normal operators which converge in operator norm. Thus A is normal. □

Since the departure from normality is difficult to calculate for a given $A \in E_w(H)$, we now give a simple upper bound.

Proposition 4.11. *Let $A \in E_w(H)$. Then*

$$\nu_w(A) \leq 2\,|A|_w\,.$$

Proof. Follows from Proposition 4.7 (ii). □

We are now able to obtain growth estimates for the resolvents of operators in a given compactness class. Before doing so we recall the bound for the resolvent of a normal operator. If D is a normal operator on a separable Hilbert space, then

$$\|R(D;z)\| = \frac{1}{d(z,\sigma(D))} \quad (\forall z \in \rho(D))\,. \tag{9}$$

The following is the main result of this section.

Theorem 4.12. *Let $A \in E_w(H)$. Then*

$$\|R(A;z)\| \leq \frac{1}{d(z,\sigma(A))} F_{\dot{\bar{w}}}\left(\frac{\nu_w(A)}{d(z,\sigma(A))}\right) \quad (\forall z \in \rho(A))\,. \tag{10}$$

Proof. Fix $z \in \rho(A)$. By Proposition 4.7, the operator A has a Schur decomposition with normal part D and quasi-nilpotent part N. Thus, we know that $\sigma(A) = \sigma(D)$, that $(zI - D)^{-1}$ exists and that $(zI - D)^{-1}N$ is quasi-nilpotent. Furthermore

$$s_k((zI - D)^{-1}N) \leq \left\|(zI - D)^{-1}\right\| s_k(N) = \frac{s_k(N)}{d(z, \sigma(D))} \leq \frac{|N|_{\dot{w}} \dot{w}_k}{d(z, \sigma(D))} = \frac{|N|_{\dot{w}} \dot{w}_k}{d(z, \sigma(A))},$$

using (9) as well as (1) and Proposition 4.7. Now $(I - (zI - D)^{-1}N)$ is invertible in $L(H)$ and, using Proposition 4.2, it follows that

$$\left\|(I - (zI - D)^{-1}N)^{-1}\right\| \leq F_{\dot{w}}\left(\frac{|N|_{\dot{w}}}{d(z, \sigma(A))}\right).$$

Since

$$(zI - A) = (zI - D)(I - (zI - D)^{-1}N),$$

we can conclude that $(zI - A)$ is invertible in $L(H)$ and

$$\|R(A; z)\| \leq \|R(D; z)\| \left\|(I - (zI - D)^{-1}N)^{-1}\right\| \leq \frac{1}{d(z, \sigma(A))} F_{\dot{w}}\left(\frac{|N|_{\dot{w}}}{d(z, \sigma(A))}\right).$$

Taking the infimum over all Schur decompositions the theorem follows. □

Remark 4.13.

(i) Another look at the above proof shows that the bound (10) also holds if we replace $\nu_w(A)$ by a larger quantity, say by the upper bound given in Proposition 4.11.

(ii) The bound (10) is optimal for normal A, as it reduces to the sharp bound (9).

5. Resolvent bounds for particular classes

As we saw in the last section, the growth of the resolvent of an operator belonging to a given compactness class E_w in the vicinity of a spectral point is, by Theorem 4.12, controlled by the behaviour of the function $F_{\dot{w}}$ at infinity. In this section we shall study the asymptotics of this function for particular compactness classes, namely the Schatten-Lorentz ideals, given by $w_k = k^{-1/p}$ with $p \in (0, \infty)$ and the exponential classes, given by $w_k = \exp(-ak^\alpha)$ with $a \in (0, \infty)$ and $\alpha \in (0, \infty)$.

Before starting with the Schatten-Lorentz ideals we briefly recall Stirling's approximation for the factorial in the form

$$\sqrt{2\pi k}\left(\frac{k}{\mathrm{e}}\right)^k \leq k! \leq \sqrt{\mathrm{e}^2 k}\left(\frac{k}{\mathrm{e}}\right)^k \quad (\forall k \in \mathbb{N}).$$

Lemma 5.1. *Let $p \in (0, \infty)$, and let $w_k = k^{-1/p}$ for $k \in \mathbb{N}$. Then the following inequalities hold:*

$$\exp\left(-\frac{1}{p\sqrt{k}}\right)\frac{\mathrm{e}^{1/p}}{k^{1/p}} \leq \bar{w}_k \leq \frac{\mathrm{e}^{1/p}}{k^{1/p}} \quad (\forall k \in \mathbb{N}), \tag{11}$$

$$\exp\left(-\frac{3}{p\sqrt{k}}\right)\frac{(2\mathrm{e})^{1/p}}{k^{1/p}} \leq \dot{w}_k \leq \frac{(2\mathrm{e})^{1/p}}{k^{1/p}} \quad (\forall k \in \mathbb{N}), \tag{12}$$

$$\exp\left(-\frac{6}{p}\sqrt{k}\right)\frac{(2\mathrm{e})^{1/p}}{(k!)^{1/p}} \leq \prod_{n=1}^{k}\dot{\bar{w}}_n \leq \frac{(2\mathrm{e})^{1/p}}{(k!)^{1/p}} \quad (\forall k \in \mathbb{N})\,. \tag{13}$$

Proof. We start with the case $p = 1$, that is, we set $w_k = k^{-1}$ and show that

$$\exp\left(-\frac{1}{\sqrt{k}}\right)\frac{\mathrm{e}}{k} \leq \bar{w}_k \leq \frac{\mathrm{e}}{k} \quad (\forall k \in \mathbb{N})\,, \tag{14}$$

$$\exp\left(-\frac{3}{\sqrt{k}}\right)\frac{2\mathrm{e}}{k} \leq \dot{\bar{w}}_k \leq \frac{2\mathrm{e}}{k} \quad (\forall k \in \mathbb{N})\,, \tag{15}$$

$$\exp\left(-6\sqrt{k}\right)\frac{2\mathrm{e}}{k!} \leq \prod_{n=1}^{k}\dot{\bar{w}}_n \leq \frac{2\mathrm{e}}{k!} \quad (\forall k \in \mathbb{N})\,. \tag{16}$$

Now, the upper bound in (14) follows from Stirling's approximation by observing that for all $k \in \mathbb{N}$ we have

$$\bar{w}_k^k = \frac{1}{k!} \leq \left(\frac{\mathrm{e}}{k}\right)^k\,.$$

For the lower bound in (14) we again use Stirling's approximation to obtain

$$\bar{w}_k^k = \frac{1}{k!} \geq \frac{1}{\sqrt{\mathrm{e}^2 k}}\left(\frac{\mathrm{e}}{k}\right)^k\,,$$

and we see that we are done if we can show that

$$\frac{1}{\sqrt{\mathrm{e}^2 k}} \geq \exp(-\sqrt{k}) \quad (\forall k \in \mathbb{N})\,. \tag{17}$$

The above, however, is true since, using the inequality $1 + x \leq \exp(x)$ which holds for all real x, we see that for all $k \in \mathbb{N}$ we have

$$\sqrt{k} \leq \exp(\sqrt{k} - 1)$$

from which

$$\sqrt{\mathrm{e}^2 k} \leq \exp(\sqrt{k})\,,$$

which implies (17).

We now turn to (15). For the upper bound we note that, for $k \in \mathbb{N}$ even, (14) implies

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k}{2}} \leq \frac{2\mathrm{e}}{k}\,,$$

while for $k \in \mathbb{N}$ odd, (14) implies

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k+1}{2}} \leq \frac{2\mathrm{e}}{k+1} \leq \frac{2\mathrm{e}}{k}\,.$$

For the lower bound we note that, for $k \in \mathbb{N}$ even, (14) implies

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k}{2}} \geq \exp\left(-\frac{\sqrt{2}}{\sqrt{k}}\right)\frac{2\mathrm{e}}{k} \geq \exp\left(-\frac{3}{\sqrt{k}}\right)\frac{2\mathrm{e}}{k}\,,$$

while for $k \in \mathbb{N}$ odd, (14) implies

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k+1}{2}} \geq \exp\left(-\frac{\sqrt{2}}{\sqrt{k+1}}\right)\frac{2\mathrm{e}}{k+1}\,,$$

and we are done if we can show that for all $k \in \mathbb{N}$ we have

$$\exp\left(-\frac{\sqrt{2}}{\sqrt{k+1}}\right)\frac{1}{k+1} \geq \exp\left(-\frac{3}{\sqrt{k}}\right)\frac{1}{k}\,,$$

which, in turn, is equivalent to

$$\left(1+\frac{1}{k}\right)\exp\left(-\frac{3}{\sqrt{k}}+\frac{\sqrt{2}}{\sqrt{k+1}}\right)\leq 1 \quad (\forall k\in\mathbb{N})\,. \tag{18}$$

The above, however, follows by observing that we have for all $k\in\mathbb{N}$

$$\begin{aligned}\left(1+\frac{1}{k}\right)\exp\left(-\frac{3}{\sqrt{k}}+\frac{\sqrt{2}}{\sqrt{k+1}}\right) &\leq \left(1+\frac{1}{k}\right)\exp\left(-\frac{1}{\sqrt{k}}\right)\\ &\leq \exp\left(\frac{1}{k}-\frac{1}{\sqrt{k}}\right)\leq \exp\left(-\frac{\sqrt{k}-1}{k}\right)\leq 1\,.\end{aligned}$$

This finishes the proof of (15).

Finally, the upper bound in (16) is obvious, while the lower one follows from

$$\prod_{n=1}^{k}\dot{w}_n\geq \exp\left(-3\sum_{n=1}^{k}\frac{1}{\sqrt{n}}\right)\frac{(2\mathrm{e})^k}{k!}\geq \exp\left(-6\sqrt{k}\right)\frac{(2\mathrm{e})^k}{k!}\,,$$

where we have used that $\sum_{n=1}^{k}n^{-1/2}\leq\int_0^k t^{-1/2}=2k^{1/2}$ for every $k\in\mathbb{N}$.

This finishes the proof of the lemma for $p=1$. The general case follows by taking p-th roots in (14), (15) and (16). □

In order to be able to study the behaviour of $F_{\dot{w}}$ we require another auxiliary result. Before stating it we introduce some more notation. If f and g are two real-valued functions defined on a neighbourhood of ∞, we write

$$f(r)\sim g(r) \text{ as } r\to\infty$$

if

$$\lim_{r\to\infty}\frac{f(r)}{g(r)}=1\,.$$

For later use, we note the following relation between the asymptotics of a function and that of its inverse.

Lemma 5.2. *Let $a,b\in(0,\infty)$ and let I and J be neighbourhoods of ∞. Suppose that $f:I\to J$ is a bijection with inverse $f^{-1}:J\to I$. Then the following assertions hold.*

(i) *If*

$$f(r)\sim ar^b \text{ as } r\to\infty$$

then

$$f^{-1}(r)\sim\left(\frac{r}{a}\right)^{1/b} \text{ as } r\to\infty\,.$$

(ii) *If*

$$\log f(r)\sim ar^b \text{ as } r\to\infty$$

then

$$f^{-1}(r)\sim\left(\frac{\log r}{a}\right)^{1/b} \text{ as } r\to\infty\,.$$

(iii) *If*

$$\log f(r) \sim a(\log r)^b \ \text{ as } r \to \infty$$

then

$$\log f^{-1}(r) \sim \left(\frac{\log r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,.$$

Proof.

(i) This follows from

$$\lim_{r\to\infty} \frac{(r/a)^{1/b}}{f^{-1}(r)} = \lim_{r\to\infty} \frac{(f(r)/a)^{1/b}}{f^{-1}(f(r))} = \left(\lim_{r\to\infty} \frac{f(r)}{ar^b}\right)^{1/b} = 1\,.$$

(ii) If

$$(\log \circ f)(r) \sim ar^b \text{ as } r \to \infty\,,$$

then by (i) we have

$$(\log \circ f)^{-1}(r) \sim \left(\frac{r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,,$$

so

$$(f^{-1} \circ \exp)(r) \sim \left(\frac{r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,,$$

hence

$$f^{-1}(r) \sim \left(\frac{\log r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,.$$

(iii) If

$$(\log \circ f)(r) \sim a(\log r)^b \text{ as } r \to \infty\,,$$

then

$$(\log \circ f \circ \exp)(r) \sim ar^b \text{ as } r \to \infty\,,$$

so by (i) we have

$$(\log \circ f \circ \exp)^{-1}(r) \sim \left(\frac{r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,,$$

hence

$$(\log \circ f^{-1} \circ \exp)(r) \sim \left(\frac{r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,,$$

whence

$$(\log \circ f^{-1})(r) \sim \left(\frac{\log r}{a}\right)^{1/b} \quad \text{as } r \to \infty\,.$$

□

We are now able to state the following result.

Lemma 5.3. *Suppose that* $p, b \in (0, \infty)$*. Let* $\Phi_p^{L,u}$ *and* $\Phi_{p,b}^{L,l}$ *be two functions given by the power series*

$$\Phi_p^{L,u}(r) = \sum_{k=0}^{\infty} \frac{1}{(k!)^{1/p}} r^k$$

$$\Phi_{p,b}^{L,l}(r) = \sum_{k=0}^{\infty} \frac{\exp(-b\sqrt{k})}{(k!)^{1/p}} r^k\,.$$

Then $\Phi_p^{L,u}$ *and* $\Phi_{p,b}^{L,l}$ *extend to entire functions with the following asymptotics*

$$\log \Phi_p^{L,u}(r) \sim \log \Phi_{p,b}^{L,l}(r) \sim \frac{1}{p} r^p \text{ as } r \to \infty .$$

Proof. Using Stirling's approximation we see that both $\Phi_p^{L,u}$ and $\Phi_{p,b}^{L,l}$ extend to entire functions. Since $\Phi_{p,b}^{L,l}(r) \leq \Phi_p^{L,u}(r)$ for all $r \in (0,\infty)$ the remaining assertions will hold if we can show that

$$\limsup_{r\to\infty} pr^{-p} \log \Phi_p^{L,u}(r) \leq 1 \tag{19}$$

and

$$\liminf_{r\to\infty} pr^{-p} \log \Phi_{p,b}^{L,l}(r) \geq 1 . \tag{20}$$

We start with (19). For $p \leq 1$ we have, using the ℓ_p-ℓ_1 inequality

$$\sum_{k=0}^{\infty} x_k \leq \left(\sum_{k=0}^{\infty} x_k^p \right)^{1/p} ,$$

which holds for all positive sequences $(x_k)_{k=0}^{\infty}$, the bound

$$\Phi_p^{L,u}(r) \leq \left(\sum_{k=0}^{\infty} \frac{r^{pk}}{k!} \right)^{1/p} = \exp\left(\frac{1}{p} r^p \right) ,$$

and (19) holds in this case. For $p > 1$ we split the sum as follows

$$\Phi_p^{L,u}(r) = \sum_{k<2\mathrm{e}r^p} \frac{r^k}{(k!)^{1/p}} + \sum_{k\geq 2\mathrm{e}r^p} \frac{r^k}{(k!)^{1/p}} .$$

In order to bound the first term we use Hölder's inequality to obtain

$$\begin{aligned}\sum_{k<2\mathrm{e}r^p} \frac{r^k}{(k!)^{1/p}} &\leq \left(\sum_{k<2\mathrm{e}r^p} \frac{r^{pk}}{k!} \right)^{1/p} \left(\sum_{k<2\mathrm{e}r^p} 1 \right)^{(p-1)/p} \\ &\leq (1+2\mathrm{e}r^p)^{(p-1)/p} \exp\left(\frac{1}{p} r^p \right) .\end{aligned}$$

For the second term, we use Stirling's approximation and obtain

$$\sum_{k\geq 2\mathrm{e}r^p} \frac{r^k}{(k!)^{1/p}} \leq \sum_{k\geq 2\mathrm{e}r^p} \left(\frac{\mathrm{e}r^p}{k} \right)^{k/p} \leq \sum_{k\geq 2\mathrm{e}r^p} 2^{-k/p} \leq 2 \cdot 2^{-(2\mathrm{e}r^p)/p} .$$

Combining these two estimates, the bound (19) follows for $p > 1$ as well.

We now turn to the proof of (20). For a given $r \geq 1$ choose $k \in \mathbb{N}$ such that

$$r^p - 1 < k \leq r^p .$$

Since all terms in the sum defining $\Phi_{p,b}^{L,l}$ are positive it follows that

$$\Phi_{p,b}^{L,l}(r) \geq \frac{\exp(-b\sqrt{k})}{(k!)^{1/p}} r^k .$$

Now

$$r^k \geq r^{r^p - 1}$$

and, using Stirling's approximation,

$$(k!)^{1/p} \leq (\mathrm{e}^2 k)^{1/(2p)} \left(\frac{k}{\mathrm{e}}\right)^{k/p} \leq (\mathrm{e}^2 r^p)^{1/(2p)} \frac{r^{r^p}}{\mathrm{e}^{\frac{1}{p} r^p}} .$$

Furthermore, we have

$$\exp(-b\sqrt{k}) \geq \exp(-br^{p/2}) .$$

Thus, combining all previous estimates and simplifying we have

$$\Phi_p^{L,u}(r) \geq \frac{\exp(-br^{p/2})}{\mathrm{e}^{1/p} r^{3/2}} \left(\frac{1}{p} r^p\right) ,$$

and the bound (20) follows. □

We are now ready to give upper and lower bounds for $F_{\dot{w}}$ as well as its asymptotics for w generating the Schatten-Lorentz ideal.

Proposition 5.4. *Let $p \in (0,\infty)$ and let $w_k = k^{-1/p}$ for $k \in \mathbb{N}$. Then for all $r > 0$ we have*

$$(1+r)\Phi^{L,l}_{p/2,12/p}\left((2\mathrm{e})^{2/p}(Cr)^2\right) \leq F_{\dot{w}}(r) \leq (1+r)\Phi^{L,u}_{p/2}\left((2\mathrm{e})^{2/p}(Cr)^2\right) \tag{21}$$

Moreover

$$\log F_{\dot{w}}(r) \sim \frac{4\mathrm{e}C^p}{p} r^p \ \text{ as } r \to \infty.$$

Proof. By Lemma 5.1 we have for all $k \in \mathbb{N}$

$$\exp\left(-\frac{12}{p}\sqrt{k}\right) \frac{(2\mathrm{e})^{2k/p}}{(k!)^{2/p}} \leq \prod_{n=1}^{k} \dot{w}_n^2 \leq \frac{(2\mathrm{e})^{2k/p}}{(k!)^{2/p}} .$$

Using the definition of $F_{\dot{w}}$ in (8) the inequalities in (21) follow, which, using Lemma 5.3, imply the remaining assertion. □

We now turn our attention to the exponential cases, which are compactness classes E_w with weights of the form $w_k = \exp(-ak^\alpha)$ with $a, \alpha \in (0,\infty)$. We start with two technical lemmas.

Lemma 5.5. *Let $a, \alpha \in (0,\infty)$, and let $w_k = \exp(-ak^\alpha)$ for $k \in \mathbb{N}$. Then there are strictly positive real constants $\bar{c}_{a,\alpha}$, $\dot{\bar{c}}_{a,\alpha}$ and $c_{a,\alpha}$ such that the following inequalities hold for every $k \in \mathbb{N}$*

$$\exp\left(-\frac{a}{\alpha+1}k^\alpha - \bar{c}_{a,\alpha}k^{\alpha-1/2}\right) \leq \bar{w}_k \leq \exp\left(-\frac{a}{\alpha+1}k^\alpha\right) , \tag{22}$$

$$\exp\left(-\frac{2^{-\alpha}a}{\alpha+1}k^\alpha - \dot{\bar{c}}_{a,\alpha}k^{\alpha-1/2}\right) \leq \dot{w}_k \leq \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}k^\alpha\right) , \tag{23}$$

$$\exp\left(-\frac{2^{-\alpha}a}{(\alpha+1)^2}k^{\alpha+1} - c_{a,\alpha}k^{\alpha+1/2}\right) \leq \prod_{n=1}^{k} \dot{w}_n \leq \exp\left(-\frac{2^{-\alpha}a}{(\alpha+1)^2}k^{\alpha+1}\right) . \tag{24}$$

Proof. We start with (22). First we note that

$$\bar{w}_k^k = \exp\left(-a\sum_{n=1}^{k} n^\alpha\right) \quad (\forall k \in \mathbb{N}) .$$

Since $\int_0^k t^\alpha\, dt \le \sum_{n=1}^k n^\alpha \le \int_0^{k+1} t^\alpha\, dt$, we have

$$\frac{1}{\alpha+1}k^{\alpha+1} \le \sum_{n=1}^{k} n^\alpha \le \frac{1}{\alpha+1}(k+1)^{\alpha+1} \quad (\forall k \in \mathbb{N})\,, \tag{25}$$

from which the upper bound of (22) readily follows, while the lower bound can be obtained by observing that there is a constant $K_1 > 0$ such that

$$\frac{(k+1)^{\alpha+1}}{k} \le k^\alpha + K_1 k^{\alpha-1/2} \quad (\forall k \in \mathbb{N})\,.$$

For the next pair of inequalities (23) we note that, using (22), we have for $k \in \mathbb{N}$ even

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k}{2}} \le \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}k^\alpha\right)\,,$$

while for $k \in \mathbb{N}$ odd

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k+1}{2}} \le \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}(k+1)^\alpha\right) \le \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}k^\alpha\right)\,,$$

and the upper bound follows. For the lower bound we note that by (22), we have for $k \in \mathbb{N}$ even

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k}{2}} \ge \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}k^\alpha - 2^{-\alpha+1/2}\bar{c}_{a,\alpha}k^{\alpha-1/2}\right)\,,$$

while for $k \in \mathbb{N}$ odd we have

$$\dot{\bar{w}}_k = \bar{w}_{\frac{k+1}{2}} \ge \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}(k+1)^\alpha - 2^{-\alpha+1/2}\bar{c}_{a,\alpha}(k+1)^{\alpha-1/2}\right)\,,$$

from which the lower bound follows for all $k \in \mathbb{N}$ by observing that for any $\beta > 0$ and any $K_2 > 0$ there is a constant $K_3 > 0$ such that

$$(k+1)^\beta + K_2(k+1)^{\beta-1/2} \le k^\beta + K_3 k^{\beta-1/2} \quad (\forall k \in \mathbb{N})\,. \tag{26}$$

Finally, using (23) and (25), the upper bound in (24) follows, since we have for all $k \in \mathbb{N}$

$$\prod_{n=1}^{k} \dot{\bar{w}}_n \le \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}\sum_{n=1}^{k} n^\alpha\right) \le \exp\left(-\frac{2^{-\alpha}a}{(\alpha+1)^2}k^{\alpha+1}\right)\,.$$

The lower bound in turn follows from

$$\prod_{n=1}^{k} \dot{\bar{w}}_n \ge \exp\left(-\frac{2^{-\alpha}a}{\alpha+1}\sum_{n=1}^{k} n^\alpha - \dot{\bar{c}}_{a,\alpha}\sum_{n=1}^{k} n^{\alpha-1/2}\right)$$
$$\ge \exp\left(-\frac{2^{-\alpha}a}{(\alpha+1)^2}(k+1)^{\alpha+1} - \frac{2\dot{\bar{c}}_{a,\alpha}}{2\alpha+1}(k+1)^{\alpha+1/2}\right)$$

and (26). □

Lemma 5.6. *Suppose that $a, \alpha, b \in (0,\infty)$. Let $\Phi^{E,u}_{a,\alpha}$ and $\Phi^{E,l}_{a,\alpha,b}$ be two functions given by the power series*

$$\Phi^{E,u}_{a,\alpha}(r) = \sum_{k=0}^{\infty} \exp(-ak^{\alpha+1})r^k\,, \tag{27}$$

$$\Phi^{E,l}_{a,\alpha,b}(r) = \sum_{k=0}^{\infty} \exp(-ak^{\alpha+1} - bk^{\alpha+1/2})r^k\,. \tag{28}$$

Then $\Phi^{E,u}_{a,\alpha}$ and $\Phi^{E,l}_{a,\alpha,b}$ extend to entire functions with the following asymptotics

$$\log \Phi^{E,u}_{a,\alpha}(r) \sim \log \Phi^{E,l}_{a,\alpha,b}(r) \sim a^{-1/\alpha}\frac{\alpha}{(\alpha+1)^{1+1/\alpha}}\left(\log r\right)^{1+1/\alpha} \quad \text{as } r\to\infty\,.$$

Proof. It is not difficult to see that both $\Phi^{E,u}_{a,\alpha}$ and $\Phi^{E,l}_{a,\alpha,b}$ extend to entire functions. As in the proof of the analogous result for the Schatten-Lorentz ideal, we note that since $\Phi^{E,l}_{a,\alpha,b}(r) \leq \Phi^{E,u}_{a,\alpha}(r)$ for all $r \in (0,\infty)$, the remaining assertions will hold if we can show that

$$\limsup_{r\to\infty} a^{1/\alpha}\frac{(\alpha+1)^{1+1/\alpha}}{\alpha}\left(\log r\right)^{-1-1/\alpha}\log \Phi^{E,u}_{a,\alpha}(r) \leq 1 \tag{29}$$

and

$$\liminf_{r\to\infty} a^{1/\alpha}\frac{(\alpha+1)^{1+1/\alpha}}{\alpha}\left(\log r\right)^{-1-1/\alpha}\log \Phi^{E,l}_{a,\alpha,b}(r) \geq 1\,. \tag{30}$$

We start with (29). Fix $r \geq 1$. Let $\mu(r)$ denote the maximal term of the series (27), that is,

$$\mu(r) = \max_{k\in\mathbb{N}}\left\{\exp(-ak^{\alpha+1})r^k\right\}\,,$$

and note that

$$\log\mu(r) \leq a^{-1/\alpha}\frac{\alpha}{(\alpha+1)^{1+1/\alpha}}\left(\log r\right)^{1+1/\alpha}\,, \tag{31}$$

which follows from a short calculation. Next, let

$$k(r) = \left(\frac{\log(2r)}{a}\right)^{1/\alpha}\,,$$

and observe that

$$\exp(-ak^{\alpha+1}) \leq (2r)^{-k} \quad (\forall k \geq k(r))\,.$$

Thus, for every $r \geq 1$ we have

$$\begin{aligned}\Phi^{E,u}_{a,\alpha}(r) &= \sum_{k<k(r)}\exp(-ak^{\alpha+1})r^k + \sum_{k\geq k(r)}\exp(-ak^{\alpha+1})r^k\\ &\leq \sum_{k<k(r)}\mu(r) + \sum_{k\geq k(r)}\frac{1}{2^k}\\ &\leq (k(r)+1)\mu(r) + \frac{1}{2^{k(r)}}\,,\end{aligned}$$

from which (29) follows.

We now turn to the proof of (30). For a given $r \geq 1$ choose $k \in \mathbb{N}$ such that

$$k \leq \left(\frac{\log r}{a(\alpha+1)}\right)^{1/\alpha} < k+1\,.$$

Since all terms in the sum defining $\Phi^{E,l}_{a,\alpha,b}$ are positive we have

$$\begin{aligned}\Phi^{E,l}_{a,\alpha,b}(r) &\geq \exp(-ak^{\alpha+1} - bk^{\alpha+1/2})r^k\\ &\geq \frac{1}{r}\exp\left(-b\left(\frac{\log r}{a(\alpha+1)}\right)^{1+1/(2\alpha)}\right)\exp\left(\frac{\alpha\left(\log r\right)^{1+1/\alpha}}{a^{1/\alpha}(\alpha+1)^{1+1/\alpha}}\right)\,,\end{aligned}$$

from which the bound (30) follows. ∎

We are now able to give upper and lower bounds as well as the precise asymptotics of $F_{\dot{w}}$ for weights generating exponential classes.

Proposition 5.7. *Let $a, \alpha \in (0,\infty)$ and let $w_k = \exp(-ak^\alpha)$ for $k \in \mathbb{N}$. Then for all $r \geq 1$ we have*

$$(1+r)\Phi^{E,l}_{a',\alpha,2c_{a,\alpha}}\left((Cr)^2\right) \leq F_{\dot{w}}(r) \leq (1+r)\Phi^{E,u}_{a',\alpha}\left((Cr)^2\right), \tag{32}$$

where $c_{a,\alpha}$ is the constant occurring in Lemma 5.5 and

$$a' = \frac{2^{1-\alpha}a}{(\alpha+1)^2}.$$

Moreover

$$\log F_{\dot{w}}(r) \sim 4\left(\frac{\alpha+1}{a}\right)^{1/\alpha} \frac{\alpha}{\alpha+1} (\log r)^{1+1/\alpha} \quad \text{as } r \to \infty.$$

Proof. The inequalities in (32) follow from Lemma 5.5 and the definition of $F_{\dot{w}}$ in (8). The remaining assertion follows from (32) and Lemma 5.6. □

6. Bounds for the spectral distance

The resolvent bounds deduced in Section 4 together with the Bauer-Fike argument which will be stated below allow us to derive the main result of this article: upper bounds for the spectral distance of two operators belonging to $E_w(H)$ expressible in terms of the distance of the two operators in operator norm and their w-departures from normality.

The formulation below is based on [Ban08, Proposition 4.1].

Theorem 6.1. *Let $A \in S_\infty(H_1, H_2)$. Suppose that there is a strictly monotonically increasing surjective function $g : [0,\infty) \to [0,\infty)$ and a positive constant K such that*

$$\left\|(zI-A)^{-1}\right\| \leq \frac{1}{K} g\left(\frac{K}{d(z,\sigma(A))}\right) \quad (\forall z \notin \sigma(A)).$$

Then, for any $B \in L(H_1, H_2)$, we have

$$\hat{d}(\sigma(B), \sigma(A)) \leq Kh\left(\frac{\|A-B\|}{K}\right).$$

Here, the function $h : [0,\infty) \to [0,\infty)$ is given by

$$h(r) = (\tilde{g}(r^{-1}))^{-1},$$

where $\tilde{g} : [0,\infty) \to [0,\infty)$ is the inverse of the function g.

Proof. Assume $B - A \neq 0$, since otherwise there is nothing to prove. We start by establishing the following statement:

$$\text{if } z \in \sigma(B), \text{ but } z \notin \sigma(A), \text{ then } \|B-A\|^{-1} \leq \left\|(zI-A)^{-1}\right\|. \tag{33}$$

This is done by contradiction. Let $z \in \sigma(B)$ and $z \notin \sigma(A)$. Assume to the contrary that

$$\left\|(zI-A)^{-1}\right\| \|B-A\| < 1.$$

Then $(I-(zI-A)^{-1}(B-A))$ is invertible. It follows that

$$(zI-B) = (zI-A)(I-(zI-A)^{-1}(B-A))$$

is invertible. Therefore $z \notin \sigma(B)$ which contradicts $z \in \sigma(B)$. Hence statement (33) holds.

In order to prove the theorem it suffices to show that if $z \in \sigma(B)$, then

$$d(z, \sigma(A)) \leq Kh\left(\frac{\|B - A\|}{K}\right).$$

Let $z \in \sigma(B)$. If $z \in \sigma(A)$, then the left-hand side of the above inequality is zero, hence there is nothing to prove. Now assume $z \notin \sigma(A)$. By (33) and the hypothesis we have

$$\frac{1}{\|B - A\|} \leq \left\|(zI - A)^{-1}\right\| \leq \frac{1}{K} g\left(\frac{K}{d(z, \sigma(A))}\right).$$

Since g is strictly monotonically increasing, so is $\tilde{g}$. Therefore

$$\tilde{g}\left(\frac{K}{\|B - A\|}\right) \leq \frac{K}{d(z, \sigma(A))},$$

and so

$$d(z, \sigma(A)) \leq \frac{K}{\tilde{g}\left(\frac{K}{\|B-A\|}\right)} = Kh\left(\frac{\|B - A\|}{K}\right),$$

as desired. □

By combining Theorems 4.12 and 6.1 we are finally able to state our spectral variation and spectral distance formulae.

Theorem 6.2. *Let $w \in \mathcal{W}$.*

(i) *If $A \in E_w(H)$ is not normal, then*

$$\hat{d}(\sigma(B), \sigma(A)) \leq \nu_w(A) H_w\left(\frac{\|A - B\|}{\nu_w(A)}\right) \quad (\forall B \in L(H)). \tag{34}$$

(ii) *If $A, B \in E_w(H)$ and neither A nor B are normal, then*

$$\text{Hdist}(\sigma(A), \sigma(B)) \leq mH_w\left(\frac{\|A - B\|}{m}\right), \tag{35}$$

where $m := \max\{\nu_w(A), \nu_w(B)\}$.

Here, the function $H_w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined by

$$H_w(r) = \frac{1}{\tilde{F}_{\dot{w}}^{-1}(\frac{1}{r})},$$

where $\tilde{F}_{\dot{w}}^{-1}$ is the inverse of $\tilde{F}_{\dot{w}} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$\tilde{F}_{\dot{w}}(r) = rF_{\dot{w}}(r),$$

and $F_{\dot{w}}$ is the function defined in (8).

Proof.

(i) By Theorem 4.12,

$$\|R(A; z)\| \leq \frac{1}{\nu_w(A)} \tilde{F}_{\dot{w}}\left(\frac{\nu_w(A)}{d(z, \sigma(A))}\right) \quad (\forall z \in \rho(A)),$$

so the assertion follows from the previous theorem.

(ii) Similarly, by Theorem 4.12 and Remark 4.13 we have

$$\|R(A;z)\| \leq \frac{1}{m}\tilde{F}_{\dot{w}}\left(\frac{m}{d(z,\sigma(A))}\right) \quad (\forall z \in \rho(A)),$$

and

$$\|R(B;z)\| \leq \frac{1}{m}\tilde{F}_{\dot{w}}\left(\frac{m}{d(z,\sigma(B))}\right) \quad (\forall z \in \rho(B)),$$

so the assertion again follows by invoking the Bauer-Fike argument.

□

Remark 6.3.

(i) Note that $\lim_{r\downarrow 0} H_w(r) = 0$, thus the bounds for the spectral variation and spectral distance become small when $\|A - B\|$ is small.

(ii) The bounds (34) and (35) remain valid if we replace $\nu_w(A)$ and $\nu_w(B)$ by a larger quantity, say by the upper bounds given in Proposition 4.11.

(iii) Combining (9) and Theorem 6.1 it follows that if A is a bounded normal operator on H, then

$$\hat{d}(\sigma(B),\sigma(A)) \leq \|A - B\| \quad (\forall B \in L(H)).$$

Moreover, by symmetry it follows from the above that if both A and B are bounded normal operators then

$$\operatorname{Hdist}(\sigma(A),\sigma(B)) \leq \|A - B\| .$$

Note that these two bounds can be thought of as limiting cases of the previous theorem, since, as is easily seen, we have for any $r \geq 0$

$$\lim_{C\downarrow 0} C H_w\left(\frac{r}{C}\right) = r .$$

It is in this respect that the bounds (34) and (35) are sharp.

Remark 6.4. In the case of the Schatten-Lorentz ideal and the exponential classes it is possible to give rather precise estimates for the behaviour of the general bound in Theorem 6.2 for two operators which are close in operator norm.

We start with the Schatten-Lorentz ideal. Let $p \in (0,\infty)$ and let $w_k = k^{-1/p}$ for $k \in \mathbb{N}$. Then Proposition 5.4 yields

$$\log \tilde{F}_{\dot{w}}(r) \sim \frac{4C^p e}{p} r^p \text{ as } r \to \infty$$

which, using Lemma 5.2, implies that

$$\tilde{F}_{\dot{w}}^{-1}(r) \sim \frac{1}{C}\left(\frac{p}{4e}\right)^{1/p} (\log r)^{1/p} \text{ as } r \to \infty,$$

which, in turn, gives

$$H_w(r) \sim C\left(\frac{4e}{p}\right)^{1/p} |\log r|^{-1/p} \text{ as } r \downarrow 0.$$

We now turn to the exponential classes. Let $a, \alpha \in (0,\infty)$ and let $w_k = \exp(-ak^\alpha)$ for $k \in \mathbb{N}$. Now, Proposition 5.7 yields

$$\log \tilde{F}_{\dot{w}}(r) \sim 4\left(\frac{\alpha+1}{a}\right)^{1/\alpha} \frac{\alpha}{\alpha+1} (\log r)^{1+1/\alpha} \text{ as } r \to \infty$$

which, using Lemma 5.2, implies that

$$\log \tilde{F}_{\dot{w}}^{-1}(r) \sim 4^{-\alpha/(\alpha+1)} \left(\frac{a}{\alpha+1}\right)^{1/(\alpha+1)} \left(\frac{\alpha+1}{\alpha}\right)^{\alpha/(\alpha+1)} (\log r)^{\alpha/(\alpha+1)} \quad \text{as } r \to \infty\,,$$

which, in turn, gives

$$\log H_w(r) \sim -4^{-\alpha/(\alpha+1)} \left(\frac{a}{\alpha+1}\right)^{1/(\alpha+1)} \left(\frac{\alpha+1}{\alpha}\right)^{\alpha/(\alpha+1)} |\log r|^{\alpha/(\alpha+1)} \quad \text{as } r \downarrow 0.$$

7. An application to inclusion regions for pseudospectra

Pseudospectra play an important role in numerical linear algebra and perturbation theory (see, for example, [Tre97, Dav07]). They are defined as follows.

Definition 7.1. Let $A \in L(H)$ and $\epsilon > 0$. The *ϵ-pseudospectrum of A* is defined by

$$\sigma_\epsilon(A) = \sigma(A) \cup \{\, z \in \rho(A) \,:\, \left\|(zI - A)^{-1}\right\| > 1/\epsilon \,\}\,. \tag{36}$$

The motivation behind this definition is the observation that for any $A \in L(H)$ and any $\epsilon > 0$ we have

$$\sigma_\epsilon(A) = \bigcup_{\substack{B \in L(H) \\ \|A-B\| < \epsilon}} \sigma(B) \tag{37}$$

as is easily seen using standard perturbation theory. In other words, the ϵ-pseudospectrum of a bounded linear operator is equal to the union of the spectra of all perturbed operators with perturbations that have norms strictly less than ϵ.

It turns out that if in the definition of the pseudospectrum (36) the strict inequality is replaced by a non-strict one, then the alternative characterisation (37) holds with the strict inequality replaced by a non-strict one. Curiously enough, this is no longer necessarily true for operators on Banach spaces (see [Sha09]).

While there exist efficient methods to compute pseudospectra of matrices (see, for example, [Tre97, Section 4], for a brief overview), the same is not true for operators on infinite-dimensional spaces, where the exact computation of pseudospectra can be a very challenging task. As an application of our resolvent bounds obtained in Section 4, we shall now provide circular inclusion regions for the pseudospectra of operators in a given compactness class.

Theorem 7.2. *Let $\epsilon > 0$.*

(i) *If $A \in L(H)$, then*

$$\{\, z \in \mathbb{C} \,:\, d(z, \sigma(A)) < \epsilon \,\} \subseteq \sigma_\epsilon(A)\,.$$

(ii) *If $A \in E_w(H)$ is not normal, then*

$$\sigma_\epsilon(A) \subseteq \left\{\, z \in \mathbb{C} \,:\, d(z, \sigma(A)) < \nu_w(A) H_w\left(\frac{\epsilon}{\nu_w(A)}\right) \right\}\,,$$

where H_w is the function defined in Theorem 6.2.

Proof.

(i) The inclusion relation follows immediately from the following lower bound for the resolvent of an operator

$$\|R(A; z)\| \geq \frac{1}{d(z, \sigma(A))}\,,$$

which in turn follows from

$$\frac{1}{d(z,\sigma(A))} = \sup_{\lambda\in\sigma(A)} |z-\lambda|^{-1} = r(R(A;z)) \leq \|R(A;z)\| .$$

(ii) By Theorem 4.12 we have the resolvent bound

$$\|R(A;z)\| \leq \frac{1}{\nu_w(A)}\tilde{F}_{\dot{w}}\left(\frac{\nu_w(A)}{d(z,\sigma(A))}\right) \quad (\forall z\in\rho(A)),$$

where $\tilde{F}_{\dot{w}}(r) = rF_{\dot{w}}(r)$.

If $z\in\sigma_\epsilon(A)$, then

$$\frac{1}{\epsilon} < \|R(A;z)\| \leq \frac{1}{\nu_w(A)}\tilde{F}_{\dot{w}}\left(\frac{\nu_w(A)}{d(z,\sigma(A))}\right),$$

and a short calculation shows that

$$d(z,\sigma(A)) < \nu_w(A)H_w\left(\frac{\epsilon}{\nu_w(A)}\right),$$

as desired.

□

Remark 7.3.

(i) Note that the inclusion (ii) above also follows from the characterisation (37) and Theorem 6.2 (i).

(ii) Note that the inclusion (ii) is sharp in the limiting case of normal A, since it reduces to

$$\sigma_\epsilon(A) = \{\, z\in\mathbb{C} : d(z,\sigma(A)) < \epsilon \,\}.$$

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