On cliques in edge-regular graphs

Leonard H. Soicher
School of Mathematical Sciences
Queen Mary University of London
Mile End Road, London E1 4NS, UK
L.H.Soicher@qmul.ac.uk

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Dedicated to the memory of Ákos Seress

Abstract

Let Γ be an edge-regular graph with given parameters (v, k, λ). We show how to apply a certain "block intersection polynomial" in two variables to determine a good upper bound on the clique number of Γ, and to obtain further information concerning the cliques S of Γ with the property that every vertex of Γ not in S is adjacent to exactly m or $m + 1$ vertices of S, for some constant $m \geq 0$. Some interesting examples are studied using computation with groups and graphs.

[Keywords: Edge-regular graph; Orbital graph; Strongly regular graph; Clique; Maximum clique; Regular clique; Quasiregular clique; Delsarte bound; Hoffman bound; Partial geometry]

1 Introduction

In this paper we present new results concerning the cliques in an edge-regular graph Γ with given parameters. We show how to apply a certain "block intersection polynomial" [5, 12] to determine a good upper bound on the clique number of Γ, and to obtain information on the cliques S of Γ with the property that every vertex of Γ not in S is adjacent to exactly m or $m+1$ vertices of S, for some constant $m \geq 0$.

Every orbital graph for a finite transitive permutation group is edge-regular, and we study some informative examples of orbital graphs using the permutation group functionality in GAP [8], to which Ákos Seress was a major contributor, together with the GAP package GRAPE [13] for computing with graphs with groups acting on them.

2 Definitions and background

All graphs in this paper are finite and undirected, with no loops and no multiple edges. A graph Γ is *edge-regular* with *parameters* (v, k, λ) if Γ has exactly v vertices, is regular of valency k, and every pair of adjacent vertices have exactly λ common neighbours. An *orbital graph* for a transitive permutation group G on a finite set Ω is a graph with vertex set Ω and edge set the G-orbit of some unordered pair $\{\alpha, \beta\}$ of distinct vertices, such that α and β are interchanged by some element of G. Such orbital graphs are edge-regular, and provide us with interesting examples. A graph Γ is *strongly regular* with *parameters* (v, k, λ, μ) if Γ is edge-regular with parameters (v, k, λ), and every pair of distinct nonadjacent vertices have exactly μ common neighbours. A *clique* in a graph Γ is a set of pairwise adjacent vertices, an *s-clique* is a clique of size s, and a *maximum clique* of Γ is a clique of the largest size in Γ. The size of a maximum clique in Γ, its *clique number*, is denoted by $\omega(\Gamma)$. The set of vertices adjacent to a vertex v in a graph Γ is denoted by $\Gamma(v)$.

For $n \geq k > 0$, the *Kneser graph* $K(n, k)$ has as vertices the k-subsets of $\{1, \ldots, n\}$, with two vertices adjacent precisely when they are disjoint. For example, $K(5, 2)$ is the Petersen graph. Observe that $K(n, k)$ is edge-regular, with parameters $\left(\binom{n}{k}, \binom{n-k}{k}, \binom{n-2k}{k}\right)$.

A *regular clique*, or more specifically, an *m-regular clique* in a graph Γ is a clique S such that every vertex of Γ not in S is adjacent to exactly m

vertices of S, for some constant $m > 0$. For example, if $n \geq 3$, then each maximum clique of $K(2n, 2)$ is $(n-2)$-regular. A *quasiregular clique*, or more specifically, an *m-quasiregular clique* in a graph Γ is a clique S of size at least 2, such that every vertex of Γ not in S is adjacent to exactly m or $m+1$ vertices of S, for some constant $m \geq 0$. For example, if $n \geq 3$, then each maximum clique of $K(2n-1, 2)$ is $(n-3)$-quasiregular, and each maximum clique of $K(3n, 3)$ is $(n-3)$-quasiregular. Note that a clique S of size at least 2 in a graph is m-regular precisely when S is both $(m-1)$-quasiregular and m-quasiregular.

Suppose that Γ is a strongly regular graph of valency $k > 0$ and having least eigenvalue σ (the eigenvalues of (the adjacency matrix of) Γ are determined by its parameters). In his famous thesis, Delsarte [6] proved that

$$\omega(\Gamma) \leq \lfloor 1 - k/\sigma \rfloor. \tag{1}$$

Moreover, if Γ is connected and not complete, then a clique S of Γ is regular if and only if $|S| = 1 - k/\sigma$ (see [4, Proposition 1.3.2(ii)]).

Delsarte's bound (1) was later generalised by A.J. Hoffman in a way that applies to regular graphs in general, but as in [9], we shall refer to the bound as the Delsarte bound when applied to strongly regular graphs. In the last section of this paper, we shall discuss an upper bound, proposed in [12], on the clique number of an edge-regular graph with given parameters (v, k, λ), which is sometimes strictly better than Delsarte's bound for a strongly regular graph with parameters (v, k, λ, μ) (when such parameters are "feasible" [3]).

In 1981, Neumaier [10] studied regular cliques in edge-regular graphs and proved the following (amongst other results):

Theorem 2.1. *Suppose Γ is a non-complete edge-regular graph having an m-regular s-clique. Then:*

1. *$\omega(\Gamma) = s$;*
2. *all regular cliques in Γ are m-regular cliques;*
3. *the regular cliques in Γ are precisely the cliques of size s.*

Neumaier pointed out that this result does not hold in general for regular graphs, but also that it is unknown whether an edge-regular graph which is not strongly regular can have a regular clique. Indeed, Neumaier [10] showed that if a vertex- and edge-transitive graph (such as an orbital graph) contains a regular clique then that graph must be strongly regular.

We shall study the situation of quasiregular cliques in edge-regular graphs, and in particular, generalise Neumaier's result above to apply to quasiregular cliques. This generalisation appears to be natural for edge-regular graphs that are not necessarily strongly regular, but does not hold for regular graphs in general.

3 The clique adjacency polynomial

We now describe our main tool.

The *clique adjacency polynomial* of an edge-regular graph Γ with parameters (v, k, λ) is:

$$C_\Gamma(x, y) = C_{v,k,\lambda}(x, y) := x(x+1)(v-y)-2xy(k-y+1)+y(y-1)(\lambda-y+2).$$

This is a special case of a block intersection polynomial [5, 12], and we shall be applying the following result, which follows from Theorem 3.2 of [5] and Theorem 1.1 of [12].

Theorem 3.1. *Let Γ be an edge-regular graph with parameters (v, k, λ), and suppose Γ has an s-clique S, with $s \geq 2$. Then:*

1. *$C_\Gamma(x, s) = \sum_{i=0}^{s}(i-x)(i-x-1)n_i$, where n_i is the number of vertices of Γ not in S adjacent to exactly i vertices in S;*
2. *$C_\Gamma(m, s) \geq 0$ for every integer m;*
3. *if m is a non-negative integer then $C_\Gamma(m, s) = 0$ if and only if S is m-quasiregular, in which case the number of vertices outside S adjacent to exactly m vertices in S is $C_\Gamma(m+1, s)/2$;*
4. *if m is a positive integer then $C_\Gamma(m-1, s) = C_\Gamma(m, s) = 0$ if and only if S is m-regular.*

For example, let Γ be the orbital graph of valency 240 for the primitive action of the Mathieu group M_{24} of degree 1771. Then Γ is an edge-regular (but not strongly regular) graph with parameters $(1771, 240, 38)$ (see [11]), and

$$C_\Gamma(x, y) = -y^3 + 2y^2x + 41y^2 - yx^2 - 483yx - 40y + 1771x^2 + 1771x.$$

We find that $C_\Gamma(2,21) = 0$ and $C_\Gamma(3,21)/2 = 630$. Thus each 21-clique S of Γ is 2-quasiregular, with just 630 vertices of Γ outside S adjacent to exactly 2 vertices of S. We used the primitive groups library in GAP and the GRAPE package to construct Γ and to determine that Γ does indeed have a 21-clique.

4 $(s-1)$-quasiregular s-cliques

We now consider the edge-regular graphs having an m-quasiregular s-clique with $m = s$ or $m = s-1$.

Let Γ be an edge-regular graph with at least one edge. Neumaier [10] proved that Γ has an s-regular s-clique if and only if Γ is a complete graph, and Γ has an $(s-1)$-regular s-clique if and only if Γ is a complete multipartite graph with exactly s parts, all of the same size. Now an s-quasiregular s-clique is s-regular, so it remains to examine the case of an edge-regular graph having an $(s-1)$-quasiregular s-clique. We have the following:

Theorem 4.1. *Let Γ be an edge-regular graph with parameters (v, k, λ), with $k > 0$. The following are equivalent:*

1. *Γ is complete multipartite;*
2. *$v = 2k - \lambda$;*
3. *every s-clique of Γ with $s \geq 2$ is $(s-1)$-quasiregular;*
4. *some s-clique of Γ with $s \geq 2$ is $(s-1)$-quasiregular.*

Proof. $(1 \implies 2)$ Straightforward.

$(2 \implies 3)$ An easy direct calculation shows that $C_{2k-\lambda,k,\lambda}(s-1, s) = 0$ for all real numbers s. Hence, by Theorem 3.1, part 3, every s-clique of Γ with $s \geq 2$ is $(s-1)$-quasiregular.

$(3 \implies 4)$ Trivial.

$(4 \implies 1)$ For this we use an argument similar to that used by Neumaier to prove that if Γ has an $(s-1)$-regular s-clique then Γ must be complete multipartite.

Suppose $s \geq 2$ and that Γ has an $(s-1)$-quasiregular s-clique S. Then every vertex not in S is nonadjacent to at most one vertex in S. Thus, if x and

y are distinct vertices of S then $\{x, y\}$ is an edge of Γ with the property that every vertex z is adjacent to at least one of x and y. In particular, $\{x, y\}$ is a 1-quasiregular clique, so $C_\Gamma(1, 2) = 0$ and so every edge of Γ is 1-quasiregular. Now define a relation $\equiv$ on the vertex-set of Γ by $v \equiv w$ if and only if $v = w$ or v and w are not adjacent. This relation is obviously reflexive and symmetric, and it is also transitive for if $v \equiv x$ and $x \equiv w$, then we cannot have $\{v, w\}$ being an edge, since $x = v$ or $x = w$ or x is non-adjacent to both v and w. Thus $\equiv$ is an equivalence relation, which shows that Γ is complete multipartite, the parts being the equivalence classes of $\equiv$ (and since Γ is regular of valency k, each part must have size $v - k$). □

5 Quasiregular cliques in edge-regular graphs that are not complete multipartite

We now generalise Theorem 2.1 of Neumaier.

Theorem 5.1. *Suppose Γ is an edge-regular graph, not complete multipartite, which has an m-quasiregular s-clique. Then for all edge-regular graphs Δ with the same parameters (v, k, λ) as Γ:*

1. *$\omega(\Delta) \leq s$, so in particular, $\omega(\Gamma) = s$;*
2. *all quasiregular cliques in Δ are m-quasiregular cliques;*
3. *the quasiregular cliques in Δ are precisely the cliques of size s (although Δ may have no cliques of size s).*

Proof. Let Δ be an edge-regular graph with parameters (v, k, λ), and let $C(x, y) = C_{v,k,\lambda}(x, y)$. Then $C(x, y) = C_\Gamma(x, y) = C_\Delta(x, y)$. By Theorem 3.1, part 3, since Γ has an m-quasiregular s-clique, $C(m, s) = 0$, so every s-clique of Δ is m-quasiregular.

Now suppose Δ has a clique L of size greater than s, and let K be a subset of L of size s. Then K is an s-clique of Δ, and so m-quasiregular, so every vertex outside K is adjacent to m or $m + 1$ vertices in K. However, this contradicts the fact that there is a vertex in $L \setminus K$ adjacent to all s vertices of K, since by Theorem 4.1, $m + 1 < s$ since Γ is not complete multipartite. Hence $\omega(\Delta) \leq s$. This proves part 1.

By part 1, each quasiregular clique of Γ is a maximum clique, which has size s. Thus, if t is an integer with $2 \leq t \leq s$ and n is a non-negative integer, then $C(n,t) = 0$ only if $t = s$, which shows that each quasiregular clique in Δ must have size s, and we have already pointed out that each s-clique in Δ is m-quasiregular. This completes the proof of parts 2 and 3. $\square$

We remark that this result does not hold if in the definition of an m-quasiregular clique "m or $m+1$" is replaced by "m or $m+2$". For example, consider the complement Δ of $K(5,2)$. Now Δ is strongly regular, with parameters $(10,6,3,4)$, has a (maximal) 3-clique $K = \{\{1,2\},\{1,3\},\{2,3\}\}$, such that every vertex not in K is adjacent to exactly 0 or 2 vertices in K, but Δ also has a (non-maximal) 3-clique $L = \{\{1,2\},\{1,3\},\{1,4\}\}$, such that every vertex not in L is adjacent to exactly 1, 2 or 3 vertices in L.

Furthermore, there are regular, but not edge-regular graphs having non-maximum quasiregular cliques. For example, consider the edge-union of the orbital graphs of valencies 5 and 10 for the primitive action of $\mathrm{P\Gamma L}(2,9)$ on 36 points. This is a regular, but not edge-regular, graph of valency 15, with clique number 6, but it has a 1-quasiregular maximal (but not maximum) 4-clique.

6 On spreads of regular cliques in edge-regular graphs

A *spread* in a graph is set of cliques forming a partition of the vertex set.

Theorem 6.1. *Let Γ be an edge-regular graph with parameters (v,k,λ). Suppose Σ is a spread of m-regular s-cliques of Γ, with $s \geq 2$, and let Δ be the graph obtained from Γ by removing the edges (but not the vertices) in the cliques in Σ. Then:*

1. *Δ is edge-regular, with parameters $(v, k-s+1, \lambda-2m+2)$;*
2. *each s-clique of Δ is $(m-1)$-quasiregular (although Δ may not have an s-clique);*
3. *if $C_\Delta(m-1,s) \neq 0$ then $\omega(\Delta) < s$;*

4. *each s-clique of Δ is $(m-1)$-regular if and only if $m \geq 2$ and $C_\Delta(m-2, s) = 0$.*

Proof. Let v be a vertex of Γ, and let v^* denote the set of vertices of Γ, other than v, contained in the unique clique in Σ containing v. Then, in Δ, $\Delta(v) = \Gamma(v) \setminus v^*$, and since $v^* \subseteq \Gamma(v)$, we have that

$$|\Delta(v)| = |\Gamma(v)| - |v^*| = k - (s-1).$$

Now suppose $\{v, w\}$ is an edge of Δ. Then v and w are in different parts of the spread Σ of m-regular cliques of Γ, and so $|\Gamma(v) \cap \Gamma(w) \cap v^*| = |\Gamma(w) \cap v^*| = m-1$. Similarly, $|\Gamma(v) \cap \Gamma(w) \cap w^*| = m-1$. Now v^* amd w^* are disjoint, and we conclude that

$$|\Delta(v) \cap \Delta(w)| = |(\Gamma(v) \cap \Gamma(w)) \setminus (v^* \cup w^*)| = \lambda - 2(m-1).$$

Hence Δ is edge-regular, with parameters $(v, k-s+1, \lambda - 2m + 2)$.

Suppose that S is an s-clique of Δ, let v be a vertex of Δ not in S, and let $T = \Gamma(v) \cap S$. Then $|T| = m$. Now suppose there are two distinct vertices $a, b \in T$, but $a, b \notin \Delta(v)$. Then $\{v, a, b\}$ is contained in the clique in Σ containing v, and so $\{a, b\}$ cannot be an edge of Δ, contradicting the fact that a and b are distinct vertices in S. Hence $|\Delta(v) \cap S| \in \{m-1, m\}$, so S is an $(m-1)$-quasiregular clique of Δ.

Suppose Δ has an s-clique S. Then S is $(m-1)$-quasiregular, and so by Theorem 3.1, part 3, $C_\Delta(m-1, s) = 0$. Hence, if $C_\Delta(m-1, s) \neq 0$ then $\omega(\Delta) < s$.

Finally, an $(m-1)$-quasiregular s-clique of Δ is $(m-1)$-regular if and only if the clique is also $(m-2)$-quasiregular, which is the case if and only if $m \geq 2$ and $C_\Delta(m-2, s) = 0$. □

This result is inspired by Haemers and Tonchev [9], who studied the existence and properties of spreads of regular cliques in strongly regular graphs. Many of these come from spreads of lines in partial geometries. A *partial geometry* is a 1-(v, k, r) design (with $k, r > 1$), whose blocks are called *lines*, such that every pair of distinct lines intersect in at most one point and for every line L and every point p not on L, there are exactly α lines containing p and intersecting L in a point, for some positive constant α. The *point graph* of a 1-design has the points of the design as vertices, with two distinct points joined by an edge precisely when they are on a common block. The study of partial geometries was initiated by Bose [2], where amongst much else he showed that the point graph Γ of a partial geometry $\mathcal{G}$ is strongly regular. Such a point graph has a regular clique (consisting of the points on a line in $\mathcal{G}$), and if $\mathcal{G}$ has a spread of lines (that is, a set of lines forming a partition of the point set), then this spread does not contain all the lines in $\mathcal{G}$, and each line not in the spread meets each line in the spread in at most one point. Hence, by Theorem 6.1, the point graph Δ of the 1-design $\mathcal{H}$ obtained from $\mathcal{G}$ by removing a spread of lines is edge-regular and contains quasiregular cliques. Furthermore, we believe that $\mathcal{H}$ should have very good statistical efficiency properties (see [1]).

We remark that a strongly regular graph Γ may have regular cliques, but no spread consisting of such cliques (for example, when Γ is the complement of $K(2n, 2)$ with $n > 2$), and even if it does have such a spread, the removal of the edges in the cliques of this spread may result in a graph having a smaller clique number (for example, when Γ is the complement of $K(4, 2)$).

7 On bounding the clique number of an edge-regular graph

Part 2 of Theorem 3.1 was applied in [12] to bound the clique number of an edge-regular graph Γ with given parameters (v, k, λ). Let $C(x, y) = C_{v,k,\lambda}(x, y)$, and let $b = b_{v,k,\lambda}$ be the least positive integer such that $C(m, b+1) < 0$ for some integer m. As pointed out in [12], such a b always exists, since as a polynomial in y, $C(x, y)$ has leading term $-y^3$, and this b is easy to determine, since for each $n = 2, 3, \ldots, v-1$, $C(x, n)$ is a quadratic polynomial in x, and we can easily check whether there is some integer m with $C(m, n) < 0$.

We know of no case of a strongly regular graph with parameters (v, k, λ, μ) where the bound $b = b_{v,k,\lambda}$ is worse than the Delsarte bound, and some cases where the bound b is strictly better (see also [12]). Somewhat trivially, since $C(0, n) = n(n-1)(\lambda - n + 2)$, we have $b \leq \lambda + 2$, but such an inequality does not always hold for the Delsarte bound.

Less trivially, the parameters with smallest v for which the existence of a strongly regular graph is unknown are $(v, k, \lambda, \mu) = (65, 32, 15, 16)$ (see [3]).

A strongly regular graph with these parameters would have least eigenvalue $(-1-\sqrt{65})/2$, and the Delsarte bound would be $8 = \lfloor 1+64/(1+\sqrt{65})\rfloor$. However, we calculate that $b_{65,32,15} = 7$, and so any edge-regular graph Δ with parameters $(65,32,15)$ has $\omega(\Delta) \le 7$. Perhaps it would be fruitful to search for a strongly regular graph with parameters $(65,32,15,16)$ and containing a clique of size 7.

As a further example, we consider the strongly regular graphs with parameters $(144,39,6,12)$. There exists such a graph, which is an orbital graph for the primitive action of the simple group $\mathrm{PSL}(3,3)$ on 144 points (see [7]). We used the primitive groups library in GAP and the GRAPE package to construct this graph and to determine that it has a maximum clique of size 4. Now any strongly regular graph with parameters $(144,39,6,12)$ has least eigenvalue -9, and so the Delsarte bound is $5 = \lfloor 16/3\rfloor$. However, $C_{144,39,6}(1,5) = -12$, and so any edge-regular graph Δ with parameters $(144,39,6)$ has $\omega(\Delta) \le 4$, and as we have seen, equality in this bound can be achieved.

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