Novel Scaling of Multiplicity Distributions in Sequential-Fragmentation and Percolation Processes

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A novel scaling for the distributions of total number of fragments (i.e., multiplicity) is found in the shattering phase of nonequilibrium, sequential-fragmentation process and in the percolation process. It is the counterpart of the Koba-Nielsen-Olesen scaling when multiplicity fluctuations are small. The relations between n-fragment cumulants and two-fragment cumulants provide easy tests to check this scaling experimentally. [S0031-9007(97)03383-8]

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Fragmentation is an ubiquitous and universal process in nature. One of the simplest and most puzzling features appearing in various experiments is the power-law behavior of the *fragment-size* (mass) *distribution:* $n(s) \sim s^{-\tau}$. By analogy with the scaling theory of phase transitions, this behavior was often interpreted as the signature of the critical behavior though a more precise analysis [1] indicates that this sign is equivocal and other pertinent observables have to be considered. One of the simplest ones is the *fragment-multiplicity distribution* (a distribution of the total number of fragments) $P(m) = \sum_s P_s(m)$, where $P_s(m)$ is the probability distribution of the number of fragments of size s. This quantity is intensely studied, e.g., in the strong interaction physics where simple behavior of much of the data on hadron multiplicity distributions [2] seems to point to a simple and perhaps deep statistical mechanism of the particle production, which can be rather independent of the particular dynamical process [3]. Some time ago Koba, Nielsen, and Olesen (KNO) suggested an asymptotic scaling of the multiplicity probability distribution in strong interaction physics [4],

$$\langle m\rangle P(m) = \Phi(z)\,, \quad z \equiv \frac{m - \langle m\rangle}{\langle m\rangle}\,, \tag{1}$$

where the asymptotic behavior is defined as $\langle m\rangle \to \infty, m \to \infty$ for a fixed $(m/\langle m\rangle)$ ratio and $\langle m\rangle$ is the multiplicity of fragments averaged over an ensemble of events. The KNO scaling means that data for differing energies (hence differing $\langle m\rangle$) should fall on the same curve when $\langle m\rangle P(m)$ is plotted against the scaled variable $m/\langle m\rangle$. More recently, the studies using the nonequilibrium and conservative fragmentation-inactivation binary (FIB) model [1,5] have shown that the KNO scaling (1) is a benchmark of the second-order shattering phase transition associated with breaking the initial mass into the "dust fragments." An asymptotic ($t \to \infty$) fragment mass distribution at this phase transition is the power law with exponent $r \le 2$ [1,5]. Natural questions appear then: (i) Is the KNO a unique asymptotic scaling of the multiplicity probability distributions, and (ii) what kind of scaling, if any, do the multiplicity distributions satisfy in the shattering phase of the FIB model where the fragment mass distribution is also a power law but with the exponent $\tau > 2$ [1,5]? The FIB model, which describes well the data both on the fragment-size (mass or charge) distribution n_s in nuclear heavy-ion fragmentation [6] as well as on the hadron (fragment) multiplicity distributions $P(m)$ in strong interaction physics [7], is well suited to investigate these two different aspects of the fragmentation process.

To address the first question, let us choose a multiplicity m with a probability distribution centered at $\nu(\langle m\rangle, x)$ and of standard deviation $\sigma(\langle m\rangle, x)$, with x distributed on $(0, \infty)$ according to some probability law $f(x)$. Two simple cases can be considered with wide fluctuations of the standard deviation, say, $\sigma^2(\langle m\rangle, x) \sim \langle m\rangle x$.

(i) The Poisson transform of $f(x)$,

$$P(m) = \int_0^\infty f(x)\exp[-\nu\langle m\rangle, x)]\,\frac{[\nu(\langle m\rangle, x)]^m}{m!}\,dx\,, \tag{2}$$

with the normalization conditions $1 = \int_0^\infty f(x)\,dx$ and $\langle m\rangle = \int_0^\infty f(x)\nu(\langle m\rangle, x)\,dx$. This corresponds to *large* fluctuations of $\nu(\langle m\rangle, x)$ since $\nu(\langle m\rangle, x) \sim \sigma^2(\langle m\rangle, x)$, and one obtains asymptotically

$$\exp(-\langle m\rangle x)\,\frac{(\langle m\rangle x)^m}{m!} \to \frac{1}{\langle m\rangle}\delta\Big(x - \frac{m}{\langle m\rangle}\Big);$$

i.e., $P(m)$ given in (2) satisfies the KNO scaling if the width of the distribution $f(x)$ is larger than the width of the Poisson distribution.

(ii) The Gauss transform of $f(x)$,

$$P(m) = \int_0^\infty f(x)\,\frac{1}{\sqrt{2\pi\langle m\rangle x}}\exp\Big(-\frac{(m - \langle m\rangle)^2}{2\langle m\rangle x}\Big)\,dx\,. \tag{3}$$

with the normalization conditions $1 = \int_0^\infty f(x)\,dx$ and $\langle m\rangle = \langle m\rangle\int_0^\infty f(x)\,dx$. In this case, fluctuations of $\nu(\langle m\rangle, x)$ are indeed *small* since $\nu(\langle m\rangle, x) \sim \langle m\rangle$ is the same for all values of x, and $P(m)$ given by (3) satisfies then the scaling law

$$\langle m\rangle^{1/2} P(m) = \Phi(z')\,, \quad z' = \frac{m - \langle m\rangle}{\langle m\rangle^{1/2}}\,, \tag{4}$$

with

$$\Phi(z') = \int_0^\infty \frac{f(x)}{\sqrt{x}} \exp\left(-\frac{z'^2}{2x}\right) dx\,. \tag{5}$$

This is the second possible form of the asymptotic scaling law of multiplicity distributions. The scaling functions in the class of Gauss transforms (5) are symmetric about $z' = 0$ though, in general, $\Phi(z')$ could be asymmetric.

In the FIB model [1,5], one deals with fragments characterized by some conserved scalar quantity that is called the *fragment mass*. The ancestor fragment of mass N is relaxing via an ordered and irreversible sequence of steps. The first step is either a binary fragmentation, $(N) \to (j) + (N - j)$, or an inactivation $(N) \to (N)^*$. In the following steps, the relaxation process continues independently for each active descendant fragment until either the low mass cutoff for monomers is reached or all fragments are inactive. For any event, the fragmentation and inactivation occur with the probabilities per unit of time $\sim F_{j,k-j}$ and $\sim I_k$, respectively. The fragmentation probability p_F without specifying the masses of the descendants is $p_F(k) = \sum_{i=1}^{k-1} F_{i,k-i}\,(I_k + \sum_{i=1}^{k-1} F_{i,k-i})^{-1}$. If instability of smaller fragments is more important than instability of larger fragments, p_F is a decreasing function of the fragment size. This is the ∞-*cluster phase*. The *transition line* is characterized by the fragment-size independence of the probability p_F (the scale-invariant branching process) at any stage of the process until the cutoff scale for monomers. Finally, when instability of larger fragments is more important than instability of smaller ones, p_F is an increasing function of the fragment size and the total mass is converted into finite-size fragments. This is the *shattered phase.*

In Fig. 1 we show examples of the asymptotic fragment multiplicity distributions in the shattering phase obtained by solving the FIB cascade equations with the rate functions: $F_{j,k-j} \equiv [j(k-j)]^\alpha$, $I_k \equiv k^\beta$ for two sets of parameters, $\alpha = -1/3$, $\beta = -1$ and $\alpha = 2/3$, $\beta = 1$, which yield asymptotically the power-law mass distribution [1] with *exactly* the same exponent $\tau = 10/3$. One may notice that for both fragmentation rate functions $F_{j,k-j}$ in Fig. 1, the scaling function is asymmetric about $z' = 0$. In general, this asymmetry increases when approaching the transition line. The strong dependence of the multiplicity distributions on the rate functions (see Fig. 1) together with the knowledge of the fragment-size distribution, allows one to constrain the choice of phenomenological rate functions and hence to learn about the nonequilibrium dynamical phase of the fragmentation. Below we will discuss examples of the multiplicity distributions in the percolation to show that the scaling law (4) has a broad range of the validity. In Fig. 2 we show the multiplicity distributions in a 3D-bond percolation on a cubic lattice of different sizes for a fixed bond activation parameter $q_{\rm CR} = 0.2488$ which corresponds to a second-order critical point in the infinite network. $\Phi(z')$ at this point is almost symmetric about $z' = 0$.

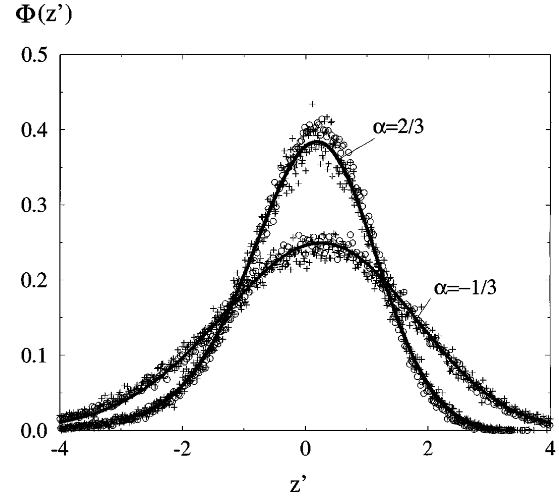

FIG. 1. Typical multiplicity distributions $[\Phi(z') \equiv \langle m\rangle^{1/2}P(m)]$ in the shattering phase of the FIB process for the rate functions $F_{j,k-j} = [j(k-j)]^\alpha$, $I_k = k^\beta$ with parameters $\alpha = -1/3$, $\beta = -1$ and $\alpha = 2/3$, $\beta = 1$ are plotted in the scaling variable $z' \equiv (m - \langle m\rangle)/\langle m\rangle^{1/2}$ (4). Open circles and crosses correspond to the Monte Carlo solutions of the multiplicity rate equations of the FIB model for initial sizes $N = 2^{11} = 2048$ and $N = 2^{13} = 8192$, respectively. Each point in these plots has been obtained for 10^6 events. The solid curve shows the Shanks extrapolation $(N \to \infty)$ of the solution of the *exact* recurrent equations of the FIB model for the system up to size $N = 1024$.

The higher order multiplicity correlations of a FIB process at the transition line have the linked-pair, hierarchical structure [8], and higher order cumulants f_p can be expressed as sums of products of linked two-particle cumulants f_2. To see the structure of the correlations associated with the scaling law (4), let us first define the

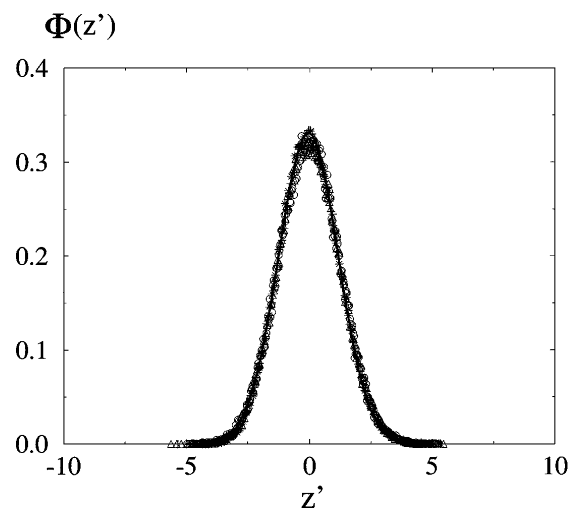

FIG. 2. The multiplicity distributions for the 3D-bond percolation on cubic lattices of different sizes $[N = 6^3$ (asterisks), 9^3 (crosses), 12^3 (triangles), and 15^3 (circles)] and for a fixed bond activation parameter $q_{\rm CR} = 0.2488$. The solid line shows the Gaussian fit of results for $N = 12^3$. Each point in this plot has been obtained for 10^5 events.

generating function $Q_N(x) = \sum_{m=0}^{\infty} P_N(m)\,(1+x)^m$, associated with the probability distribution $P_N(m)$ for a finite value N of the initial mass [9]. With the hypothesis of the scaling (4), one gets in the limit of large $\langle m\rangle$ (i.e., large N),

$$Q(x) = (1+x)^{\langle m\rangle} \int_{-\infty}^{\infty} \Phi(z') \exp[z' \langle m\rangle^{1/2} \ln(1+x)]\, dz' = (1+x)^{\langle m\rangle} \sum_{l=0}^{\infty} \mu_l \frac{[\langle m\rangle^{1/2} \ln(1+x)]^{\ell}}{l!}\,, \quad (6)$$

where $\mu_l \equiv \langle z'^l\rangle$ are the normalized moments of the scaling function $\Phi(z')$ (4) which are free asymptotically from the redundant dependence on the average multiplicity $\langle m\rangle$. Equation (6) allows one to express moments μ_l in terms of the factorial cumulant moments f_p and in this way to find the relation between $\{\mu_l\}$ moments and integrals of the cumulant correlation functions. In particular,

$$\begin{aligned}
\mu_0 &= 1, \quad \mu_1 = 0\,, \\
\mu_2 &= \gamma_2 \langle m\rangle + 1\,, \\
\mu_3 &= \gamma_3 \langle m\rangle^{3/2} + \left[\frac{1 - 2\gamma_2\langle m\rangle}{\langle m\rangle^{1/2}}\right], \quad (7)\\
\mu_4 &= \gamma_4 \langle m\rangle^2 + 3(\gamma_2\langle m\rangle + 1)^2 \\
&\quad + \left[\frac{6\gamma_3\langle m\rangle^2 - 23\gamma_2\langle m\rangle + 1}{24\langle m\rangle}\right], \\
&\vdots
\end{aligned}$$

γ_p in the above formula are the normalized cumulant moments, $f_p/(f_1)^p$. Different terms in the squared brackets vanish when N becomes infinite. Note that in the Gaussian case, $\gamma_p = 0$ ($p \geq 3$) asymptotically and, consequently, $\mu_3 = 0$, $\mu_4 = 3(\gamma_2\langle m\rangle + 1)^2$, and so on. This is because the generating function in this case is $\ln[Q(x)] = L \ln[1 + (1 - \mu_2 x)]$ and $\gamma_p \sim \langle m\rangle^{1-p}$ for $p \geq 3$. For large multiplicities Eqs. (7) can be rewritten as relations between normalized factorial cumulant moments,

$$\gamma_p = \tilde{A}_p(\sqrt{\gamma_2}\,)^p = \tilde{A}_p(\sqrt{f_2} f_1^{-1})^p \quad (p = 3, 4, \ldots)\,, \quad (8)$$

with the $\langle m\rangle$-independent hierarchical amplitudes $\tilde{A}_p$, which measure the amplification of higher order correlations and provide a unique characterization of any fragmentation model obeying the scaling law (4) in any of its multiplicity domains.

We have seen in Fig. 2 that the multiplicity probability distribution in the 3D percolation at $q = q_{\mathrm{CR}}$ is well approximated by the distribution in the class of Gauss transforms. At this point, both $\gamma_2\langle m\rangle$ ($\sim\mu_2$) and $\gamma_3\langle m\rangle^2$ are approximately constant, independently of the size of the network. To see the validity of this scaling also outside of the critical point, in Fig. 3 we plot $\gamma_2\langle m\rangle$ and $\gamma_3\langle m\rangle^2$ vs $\langle m\rangle/N$ for different lattice sizes of the 2D and 3D bond percolation. Each point in Fig. 3 corresponds to the calculation for a *fixed* value of the bond activation probability q. As seen from these plots, the results for percolation networks of different sizes are superposing well not only around the critical point but also outside of it and, in particular, for $q < q_{\mathrm{CR}}$. We have checked that these results are *quantitatively* unchanged when solving percolation on various types of lattices. This confirms the universal and robust character of these correlations and makes them suitable for the phenomenological analysis of various fragmentation processes in nature. The maximum of μ_2 is found at the value of the reduced multiplicity $\langle m\rangle/N$ corresponding *exactly* to the value at the critical point [10] both in 2D and 3D percolation. We have checked that the same features hold for $D > 3$.

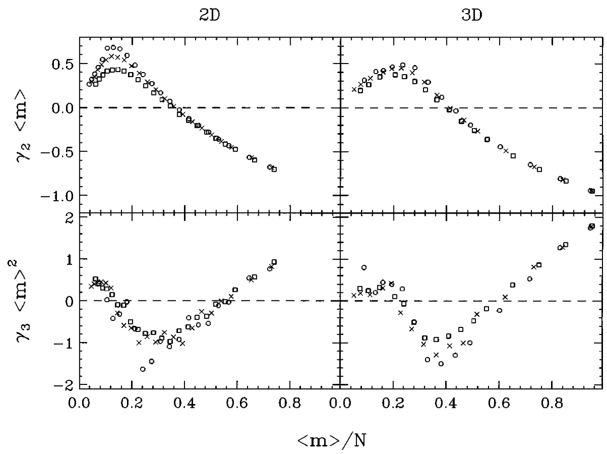

FIG. 3. $\gamma_2\langle m\rangle$ ($\sim\mu_2$) and $\gamma_3\langle m\rangle^2$ are plotted versus the reduced multiplicity $\langle m\rangle/N$ for percolation networks of different sizes in the 2D square lattice [$N = 15^2$ (squares), 30^2 (crosses) and 90^2 (circles)] and in the 3D cubic lattice [$N = 6^3$ (squares), 10^3 (crosses), and 20^3 (circles)]. Each calculation corresponds to 10^5 events.

For the experimental verification of the scaling laws (1) and (4), the scaling function [$\Phi(z)$ or $\Phi(z')$] or its moments, as well as the value of $\langle m\rangle$ (or $\langle m\rangle/N$), have to be calculated always for a fixed given value of the "control parameter" in the studied process. Data obtained in this way for different values of this parameter can then be compared. In the above studied case of percolation, for a given size N of the network, the evolution of the fragmentation process was controlled by the bond activation parameter. In the strong interaction physics, this is a total energy in the center of mass [2] for a given reaction. For processes like the fragmentation in evolving porous media [11] or the cluster-cluster aggregation [12], the appropriate control parameter is the mean fragment size which is related to the elapsed time in these processes.

In conclusion, we have demonstrated an existence of the new asymptotic scaling law (4) for distributions of the total number of fragments (i.e., multiplicity), which includes, as a special limit, the typical thermodynamic system such as the percolation. In view of the generality of the models considered, one is tempted to conjecture that there exist two and only two scaling laws. (i) The KNO scaling law (1) associated with large multiplicity fluctuations at the shattering phase transition, and (ii) the new scaling law (4) associated with small multiplicity fluctuations. In the latter case, the scaling law and the corresponding correlation coupling scheme (8) hold in the whole shattering phase of the FIB process and in the percolation. Similarly as for the KNO problem, all essential information about the multiplicity distributions are reducible to the form of the scaling function $\Phi(z')$ which is a benchmark of the fragmentation process, allowing one not only to distinguish between different fragmentation models but also, for a given model, between different fragmentation domains. This important function can be studied phenomenologically using the new family of $\{\mu_l\}$ moments or, equivalent, using the relations (7) and (8) for factorial cumulant moments. The values of moments μ_k and the corresponding hierarchical amplitudes $\tilde{A}_k$ for different reduced multiplicities $\langle m\rangle/N$, together with the moments of the fragment-size distribution n_s provide the two independent and complementary pieces of information characterizing the fragmentation process.

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